## Chapter 4 : Linear Transformation

### (4.1) Definitions & Examples

**Def:** Let $V, W$ be two vector spaces, then mapping (function) $L: V \longrightarrow W$ is called a linear transformation, if:

1) $L(v_1 + v_2) = L(v_1) + L(v_2)$ , $\forall v_1, v_2 \in V$

2) $L(\alpha v) = \alpha L(v)$ , $\alpha \in \mathbb{R}$ , $v \in V$

note: we can replace the above 2 conditions by a single condition, which is:

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$$

ex 1: $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ , defined by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ .

Show that $L$ is a linear transformation.

proof: 1) let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ , then:

$$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} y_1 \\ 0 \end{pmatrix} = L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + L\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

So, $L(v_1 + v_2) = L(v_1) + L(v_2)$

2) $L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

$\alpha\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

So, $L(\alpha v) = \alpha L(v)$

$\Longrightarrow$ L is a linear transformation.

note: $L: V \longrightarrow V$ is called a linear operator.

ex2: $L: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + 1$

Is L a linear transformation?

$L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \alpha x_1 + 1$

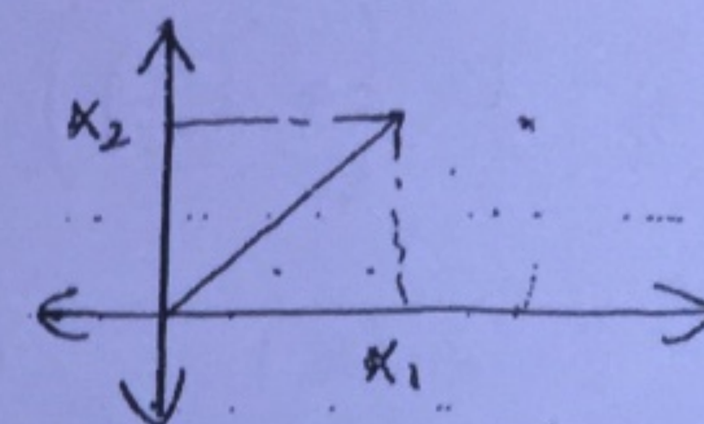
$\alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha(x_1 + 1) = \alpha x_1 + \alpha$

$L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \neq \alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

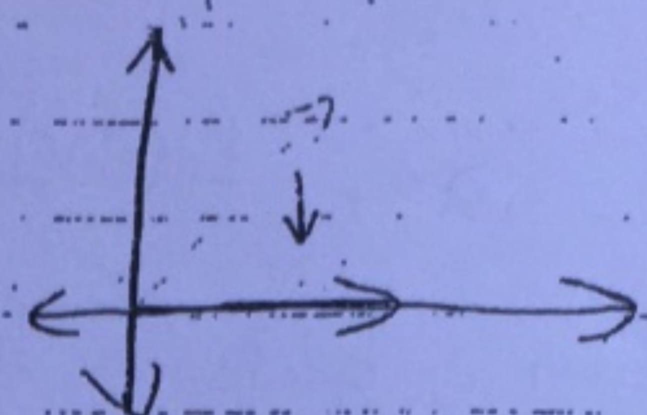
So, L isn't a linear transformation.

Remark: any vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ can be represented geometrically as follows:

$V = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow$




ex: $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \Longrightarrow$



(progection on x-axis)

ex3: (Identity Operator):

$L : V \longrightarrow V$ , $L(v) = v$

Is L a linear transformation?

1) $L(\alpha v) = \alpha v = \alpha \cdot L(v)$

2) $L(v_1 + v_2) = v_1 + v_2 = L(v_1) + L(v_2)$

So, L is a linear trasformation.

*Properities of Linear Transformation:

1) $L(0_v) = 0_w$

2) $L(-v) = -L(v)$

3) $L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n)$

ex 4: $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 4 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

1) $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = 4\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$

$4\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = 4\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = 4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 4\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + L\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

So, $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + L\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$

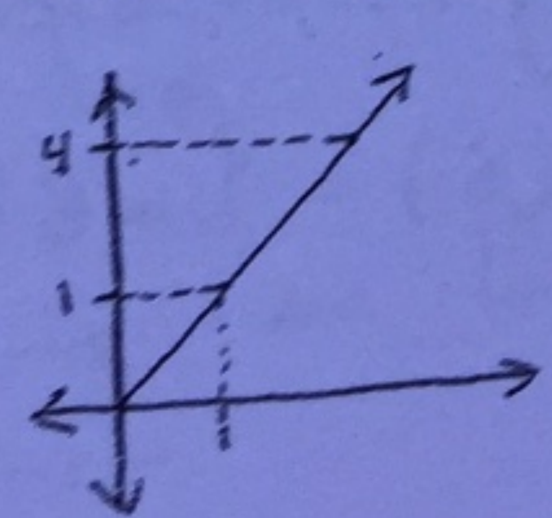
2) $L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = 4\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = 4\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

So, $L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \alpha\, L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$.

$\Longrightarrow$ L is a linear trasformation.

note: in this example, L represents a stretching by 4 in $\mathbb{R}^2$.



ex 8: $L: \mathbb{R}^2 \longrightarrow P_2$ , $L\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a + bz$

Is $L$ a linear trasformation?

$$L\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$$

$$= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) z$$

$$= \alpha(x_1 + x_2 z) + \beta(y_1 + y_2 z)$$

$$= \alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta L\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

So, $L$ is a linear transformation.

Def: Let $L: V \longrightarrow W$ be a linear transformation, then:

1) The kernal of $L$, denoted by $\mathrm{Ker}(L)$ is a subspace of $V$ defined by:

$$\mathrm{Ker}(L) = \{ v \in V \mid L(v) = 0_W \}$$

2) If $S$ is a subspace of $V$, then the image of $S$, denoted by $L(S)$ is a subspace of $W$, defined as:

$$L(S) = \{ L(x) \mid x \in S \}$$

or

$$L(S) = \{ w \in W \mid L(x) = w \text{ for some } x \in S \}$$

3) If $S = V$, the image of $V$, denoted by $L(V)$ is called the range of $L$. (Range of $L$).

4) If $L(V) = (W)$, then $L$ is called onto (epic).

5) $L : V \longrightarrow W$ is called (1-1), iff:
$L(v_1) = L(v_2) \Longrightarrow v_1 = v_2$

Th: $L : V \longrightarrow W$ is (1-1) iff $Ker(L) = \{0\}$.

ex 6: Consider the L.T, $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ define by:

$$L\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$$

Find $Ker(L)$, then find a basis of $Ker(L)$ and $\dim(Ker(L))$.

1) $Ker(L) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid L\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} b \\ b \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\Longrightarrow b = 0$

So, $Ker(L) = \left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} ; a, c \in \mathbb{R} \right\}$

2) $\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a e_1 + c e_3 ; = span\{$

Since $e_1, e_3$ are L.I & span $(Ker(L))$, the

$\therefore$ a basis for $Ker(L)$ is $\{e_1, e_3\}$

3) $\dim(Ker(L)) = 2$ ($L$ isn't (1-1), because $Ker(L) \neq \{0\}$).

4) Range of $L$ & its basis :

~~to find~~

$$L\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$$

$$Range(L) = \left\{ L\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} = \left\{ \begin{pmatrix} b \\ b \\ b \end{pmatrix} \right\} ; b \in \mathbb{R}$$

$$= \left\{ b\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$= span\left( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

So, a basis for $Range(L)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\dim(Range(L)) = 1$

Note that $L$ isn't onto, because $L(\mathbb{R}^3) = Range(L) \neq \mathbb{R}^3$

note: $L: V \longrightarrow W$ is L.T. of $V$ is finite dimensioned, then $\dim(Ker(L)) + \dim(Range(L)) = \dim(V)$

ex7: If $S = \text{span}(e_2)$, find $L(S)$, $L\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$

$L(S) = \{ L(\alpha) ; \alpha \in S \}$.

$S = \text{span}(e_2) = \alpha e_2 = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}$

$L(S) = \left\{ L\begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \right\} \Rightarrow \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

So, a basis for $L(S)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\dim(L(S)) = 1$.

ex 8: If $S = \text{span}(e_1)$, find $L(S)$, $L\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$

$L(S) = \{ L(\alpha) ; \alpha \in S \}$

$S = \text{span}(e_1) = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$

$L(S) = \left\{ L\left(\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}\right) \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\dim(L(S)) = 0$

ex9) $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$$

Find Range (L), Ker (L) and check if onto & (1-1).

$$Ker(L) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ; \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$x_1 + x_2 = 0$
$x_2 + - x_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$x_1, x_2$ are leading variables.
$x_3$ is a free variable.

let $x_3 = t$, $t \in \mathbb{R}$

$x_1 = -t$, $x_2 = t$

$$Ker(L) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} \right\}$$

$$= \left\{ t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} = span\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

So, a basis for Ker(L) is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$\text{Range}(L) = L(\mathbb{R}^3)$$

$$= \left\{ \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \end{pmatrix} \right\}$$

So, $\text{Range}(L) = \mathbb{R}^2$, because:

$$\dim(\text{Ker}(L)) + \dim(\overset{\text{Range}(L)}{\cancel{\text{Ker}(L)}}) = \dim(\mathbb{R}^3)$$

$$1 + \dim(\text{Range}(L)) = 3$$

$$\Longrightarrow \dim(\text{Range}(L)) = 2$$

But Range(L) is a subspace of $\mathbb{R}^2$

$\Longrightarrow$ $\text{Range}(L) = \mathbb{R}^2$

ex10: $L : P_3 \longrightarrow P_3$

$L(p(x)) = x\,p(0) + p(1)$

find $Ker(L)$, $Range(L)$.

$L(ax^2+bx+c) = xc + a+b+c$

$Ker(L) = \left\{ ax^2+bx+c \,;\, xc+a+b+c = 0 \right\}$

$cx + (a+b+c) = 0 \cdot x + 0$

$\Rightarrow c=0$ , $a+b+c=0$

$\Rightarrow a+b=0$

$\Rightarrow b=-a$

So, $Ker(L) = \left\{ ax^2 - ax \right\}$

$ax^2 - ax = a(x^2-x)$

$= span(x^2-x)$

A basis for $Ker(L)$ is $\{x^2-x\}$

$\dim Ker(L) = 1$ (not one-to-one)

$$\begin{aligned}
\text{Range}(L) = L(P_3) &= xc + a + b + c \\
&= c(x) + (a+b+c)(1) \\
&= \alpha_1(x) + \alpha_2(1) \\
&= \text{span}(1, x) = P_2
\end{aligned}$$

$\dim(\text{Range}(L)) = 2$

Basis of $L(P_3)$ is $\{1, x\}$.

$\} = 0$

## (4.2) Matrix Representation for Linear Transformation

$\star$ If $A_{m\times n}$, we can define a L.T $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by: $L(x) = A_{m\times n} x_{n\times 1}$, $x \in \mathbb{R}^n$

proof: We need to show that $L$ is L.T.

$$\begin{aligned} L(\alpha x + \beta y) &= A(\alpha x + \beta y) \\ &= A(\alpha x) + A(\beta y) \\ &= \alpha (Ax) + \beta (Ay) \\ &= \alpha L(x) + \beta L(y) \end{aligned}$$

So, $L$ is L.T.

ex: take $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ define L.T for $A$.

$L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$L(x) = Ax$

$$L\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 2a + 3b \\ 4a + 5b \\ 6a + 7b \end{pmatrix}$$

Q: $L: V \longrightarrow W$ a L.T

$E = \{v_1, v_2, \dots, v_n\}$ is a basis for $V$

$F = \{w_1, w_2, \dots, w_n\}$ is a basis for $W$

Can we find a matrix $A_{m\times n}$ that represents $L$?

$A$ represents $L$, means $[L(v)]_F = A\,[v]_E$, $\forall v \in V$

The matrix $A$.

$$A = \left( [L(v_1)]_F \quad [L(v_2)]_F \quad \cdots \quad [L(v_n)]_F \right)$$

note: $A$ is called the matrix repesentation of $L$ corresponding to the bases $E$ & $F$.

To prove so, take a vector $v \in V$;

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$L(v) = L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$L(v) = \alpha L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n)$$

$$[L(v)]_F = [\alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n)]_F$$

$$= \alpha_1 [L(v_1)]_F + \alpha_2 [L(v_2)]_F + \cdots + \alpha_n [L(v_n)]_F$$

$$= \left( [L(v_1)]_F \quad [L(v_2)]_F \cdots [L(v_n)]_F \right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$[L(v)]_F = A\,[v]_E$$

ex: Let $L: \mathbb{R}^2 \longrightarrow P_2$ be a linear transformation defined by:

$$L\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = ax + b$$

Also, Let $E = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$, $F = \{x-1, x+1\}$

be two bases for $\mathbb{R}^2$ & $P_2$ respectively.

1) Find the matrix representation of L with respect to E & F.

$$A_{2\times2} = \left( \left[ L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \right]_F \quad \left[ L\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \right]_F \right)$$

$$L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = x+1 = 0(x-1) + 1(x+1)$$

$$\left[ L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \right]_F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = x+2 = c_1(x-1) + c_2(x+1)$$

$$x+2 = (c_1+c_2)x + (c_2 - c_1)$$

$$\Rightarrow \quad c_1 + c_2 = 1$$

$$c_2 - c_1 = 2$$

$$2c_2 = 3 \Rightarrow \boxed{c_2 = \frac{3}{2}}$$

$$c_2 - c_1 = 2 \Longrightarrow \boxed{c_1 = \frac{-1}{2}}$$

$$\left[ L\left(\binom{1}{2}\right) \right]_F = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix}$$

$$\Longrightarrow A = \begin{pmatrix} 0 & \frac{-1}{2} \\ 1 & \frac{3}{2} \end{pmatrix}$$

$$\Longrightarrow [L(v)]_F = A\,[v]_E \,,\ \forall v \in \mathbb{R}^2$$

2) Use (1) to find $L\left(\binom{2}{3}\right)$

$$\left[ L\left(\binom{2}{3}\right) \right]_F = A \left[ \binom{2}{3} \right]_E$$

$$\binom{2}{3} = 1 \binom{1}{1} + 1 \binom{1}{2}$$

$$\Longrightarrow \left[ \binom{2}{3} \right]_E = \binom{1}{1}$$

$$\Longrightarrow \left[ L\left(\binom{2}{3}\right) \right]_F = \begin{bmatrix} 0 & \frac{-1}{2} \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{5}{2} \end{bmatrix}$$

$$L\left(\begin{pmatrix}2\\3\end{pmatrix}\right) = \frac{-1}{2}(x-1) + \frac{5}{2}(x+1)$$

$$= \frac{-1}{2}x + \frac{1}{2} + \frac{5}{2}x + \frac{5}{2}$$

$$= 2x + 3$$

$*$ A special case:

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$E = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$

$F = \{e_1, e_2, \ldots, e_m\}$ is a basis for $\mathbb{R}^m$

$$A = \left( [L(e_1)]_F \quad [L(e_2)]_F \quad \cdots \quad [L(e_n)]_F \right)$$

$$\cancel{\Rightarrow A = \left( u^{-1} L(v_1) \quad u^{-1} L(v_2) \quad \cdots \quad u^{-1} L(v_n) \right)}$$

$$\Rightarrow A = \left( L(e_1) \quad L(e_2) \quad \cdots \quad L(e_n) \right).$$

So, $[L(v)]_F = A\,[v]_E$

$$\Rightarrow \boxed{L(v) = Av}$$

$*$ $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Basis $E = \{v_1, v_2, \ldots, v_n\}$ $\longrightarrow$ Basis $F = \{u_1, u_2, \ldots, u_n\}$

What is the matrix representation of $L$ with respect to the bases $E$ & $F$?

$$A = \left( [L(v_1)]_F \quad [L(v_2)]_F \quad \cdots \quad [L(v_n)]_F \right)$$

$$\Rightarrow A = \left( u^{-1} L(v_1) \quad u^{-1} L(v_2) \quad \cdots \quad u^{-1} L(v_n) \right)$$

where $u = \left( u_1 \; u_2 \; \cdots \; u_m \right)$, which is non-singular.

proof: $L(v_1) = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$

$$= (u_1 \; u_2 \; \cdots \; u_m) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

$$L(v_1) = u \, [L(v_1)]_F$$

$$\Longrightarrow [L(v_1)]_F = u^{-1} L(v_1)$$

Similarly, $[L(v_j)]_F = u^{-1} L(v_j)$, $\forall j = 1, 2, \ldots, n$

$$[L(v)]_F = A \, [v]_E$$

So, to find $A$ for $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$E = \{v_1, \ldots, v_n\}$ $\quad$ $F = \{u_1, \ldots, u_m\}$

Apply Gauss Jordan Elimination

ex: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ x-y \end{pmatrix}$$

$E = \{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{v_2} \}$ is a basis for $\mathbb{R}^2$

$F = \{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{u_3} \}$ is a basis for $\mathbb{R}^3$

1) Find the matrix representation of L corresponding to E & F.

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & & \\ 0 & 1 & 1 & L(v_1) & L(v_2) \\ 0 & 0 & 1 & & \end{array}\right]$$

$$L(v_1) = L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$L(v_2) = L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\underbrace{\phantom{xxxx}}_{I} \qquad \underbrace{\phantom{xxxx}}_{A}$$

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = A\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_E$$

2) Find $L\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right)$ using $A$.

$$\left[L\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right)\right]_F = A\left[\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right]_E$$

But $E = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\Longrightarrow c_1 = 1, \; c_2 = -1$$

$$\Longrightarrow \left[L\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right)\right]_F = \begin{pmatrix} 0 & -1 \\ 1 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$L\left(\begin{pmatrix}-1\\-2\end{pmatrix}\right) = 1\cdot u_1 - 3\,u_2 + 1\cdot u_3$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-2\\1\end{pmatrix}$$

To check that:

$$L\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x\\y\\x-y\end{pmatrix}$$

$$L\left(\begin{pmatrix}-1\\-2\end{pmatrix}\right) = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$$