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MATH 234

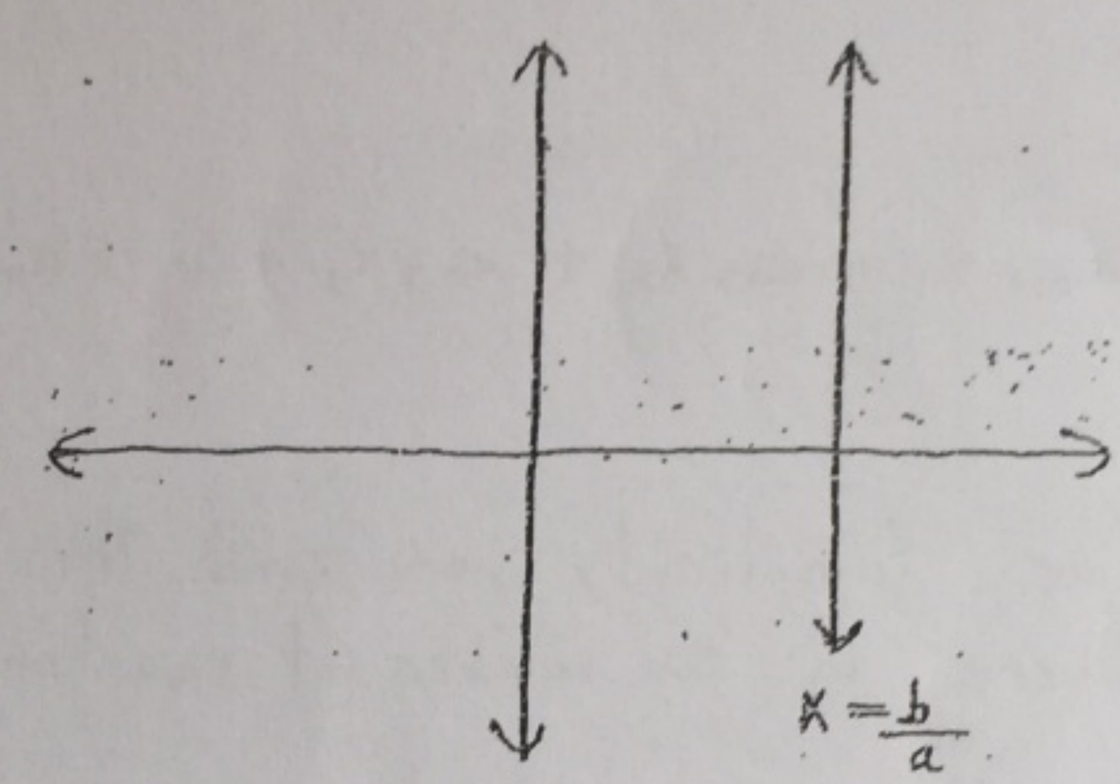
Chapter 1: Matrices and Systems of Equations

(1.1) Systems of Linear Equations

* Linear equation of one variable:

$ax = b$; a, b : constants (real numbers or scalars)
; x : variable (unknown)

$x = \frac{b}{a}$



* Linear equations in two variables:

$a x_1 + b x_2 = c$; a, b, c : constants
; x_1, x_2 : variables

$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n$

$a_1, a_2, a_3, \dots, a_n$: constants
$x_1, x_2, x_3, \dots, x_n$: variables

* A linear system of m equations and n variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m .$$

For Simplicity, we call this system $m * n$ system, where m: the number of equations, n: the number of unknowns.

ex1:
$$2x_1 + 3x_2 = 5$$
$$4x_1 - 3x_2 = 0.$$

This is a $(2 * 2)$ system.

ex 2:
$$4x_1 + 3x_2 + x_3 - x_4 = 1$$
$$5x_1 + 2x_2 + 2x_3 + 2x_4 = 2$$
$$x_2 + x_4 = 0$$

This is a $(3 * 4)$ system.

Our goal is to solve any $(m * n)$ system.

A solution for a system is a system:

$s_1, s_2, s_3, \dots, s_n$ of numbers that satisfies all the equations.

ex 1| $x_1 + x_2 = 2$ --- (1) ; solve this system

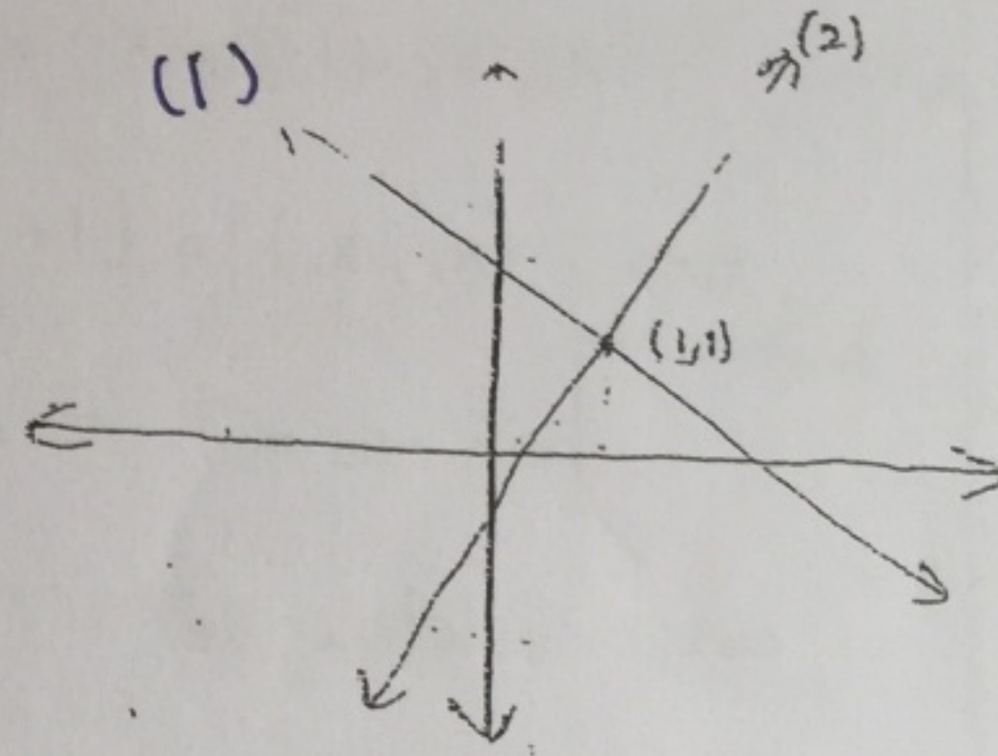
$2x_1 - x_2 = 1$ --- (2)

by adding (1) to (2):

$3x_1 = 3 \Rightarrow \boxed{x_1 = 1}$

$2(1) - x_2 = 1$ --- (2)

$\Rightarrow \boxed{x_2 = 1}$



$(x_1, x_2) = (1,1) \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

exactly one solution (unique solution) one solution

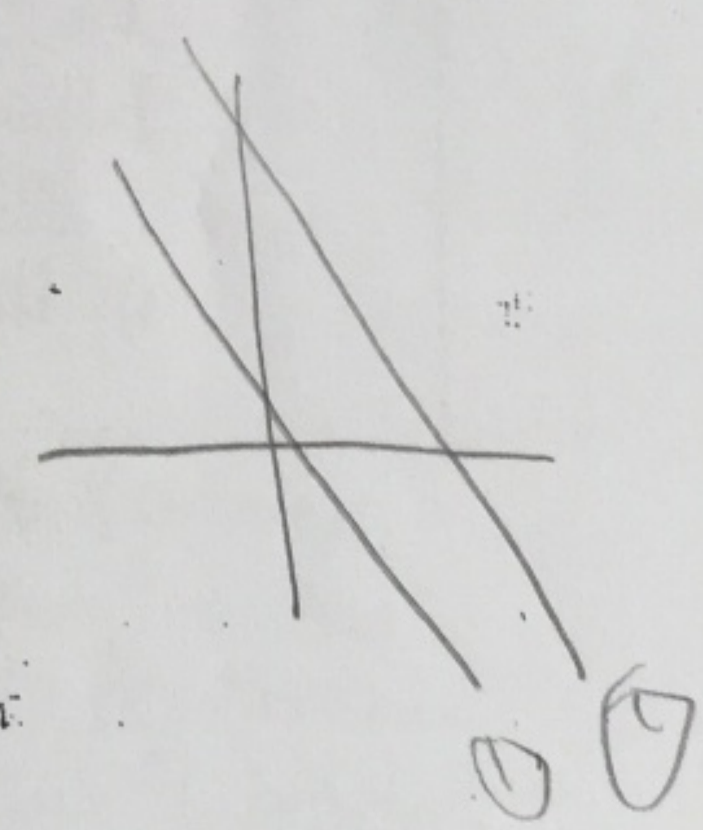
ex2: $x_1 + x_2 = 2$ --- (1) . solve this system

$x_1 + x_2 = 4$ --- (2)

subtracting (2) from (1):

$0 + 0 = -2$ (impossible) no solution

$\Rightarrow$ This system has no solution.



ex3: $x_1 + x_2 = 1$ --- (1) solve this system

$2x_1 + 2x_2 = 2$ --- (2)

$$\begin{array}{rl} & 2x_1 + 2x_2 = 2 \quad \text{--- (1)} \\ - & 2x_1 + 2x_2 = 2 \quad \text{--- (2)} \\ \hline & 0 + 0 = 0 \end{array}$$

$0 + 0 = 0 \Rightarrow$ always true many solution

$\Rightarrow$ This system has infinite number of solutions in the previous example:

$$x_1 + x_2 = 1 \Rightarrow x_1 = 1 - x_2$$

$$\Rightarrow (x_1, x_2) = (1 - x_2, x_2) \Rightarrow \begin{pmatrix} 1 - x_2 \\ x_2 \end{pmatrix}$$

let $x_2 = t \in \mathbb{R}$.

$\Rightarrow$ solution set: $\left\{ \begin{pmatrix} 1-t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$

so if $t = 0 \Rightarrow \begin{pmatrix} 1-0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow (1, 0)$

and if $t = 10 \Rightarrow \begin{pmatrix} 1-10 \\ 10 \end{pmatrix} = \begin{pmatrix} -9 \\ 10 \end{pmatrix} \Rightarrow (-9, 10)$

In general any $(m \times n)$ system has one of the following three cases:

1) Has a unique solution.
2) Has infinitely many solutions.] consistant system
3) Has no solutions.] Inconsistent system

How to solve a system?

Answer: Transform this system to an equivalent system easier to solve.

Def: Two systems of linear equations are called equivalent if:

1) They have the same variables.
2) They have the same solution

ex:

$x_1 + x_2 = 1$ --- (1)
$x_1 + 2x_2 = 2$ --- (2)

(2 * 2) system

subtracting (1) from (2):

$\Longrightarrow \boxed{x_2 = 1}$

$x_1 + 1 = 1$ --- (1)
$\Longrightarrow \boxed{x_1 = 0}$

$x_1 + x_2 = 1$ --- (1)
$3x_2 = 3$ --- (2)

Back Substution

(2 * 2) system

from (2) $\Longrightarrow \boxed{x_2 = 1}$

$x_1 + 1 = 1$ --- (1)
$\Longrightarrow \boxed{x_1 = 0}$

note: the method of solving the second equation then getting back to the first one is called "Back Substution".

The two systems are eqivalent.

Strict Triangular Substution:

ex:

easy to solve using "Back Substation".

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \quad &---(1) \\ 2x_2 + 3x_3 &= 2 \quad &---(2) \\ 4x_3 &= 0 \quad &---(3) \end{aligned}$$

3)--- $4x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

2)--- $2x_2 + 3(0) = 2 \Rightarrow \boxed{x_2 = 1}$

1)--- $x_1 + (1) + (0) = 2 \Rightarrow \boxed{x_1 = 1}$

$\Rightarrow (x_1, x_2, x_3) = (1, 1, 0)$ {unique solution}

* Matrix representation of linear systems.

ex: $x_1 + x_2 + x_3 = 4$
$x_1 - x_2 - x_3 = 5$ (2*3) system.

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ (matrix of cofficents)

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 5 \end{array}\right]$ (augmented matrix)

There are three types of $\underset{(m \times n)}{5*5}$ systems:

1) Square System, when $m=n$.

2) Over Determined System, when $m>n$.

3) Under Determined Sytem, when $m<n$.

How to transform a system?

Answer: Using elementary row operations:

1) Interchange two rows. (equations)
2) Multiply a row by a non-zero constant.
3) Replace a row by its sum with a multiple of another row.

ex on point (3):

$$x_1 + x_2 = 4 \quad ---(1)$$
$$x_1 - x_2 = 5 \quad ---(2)$$

equation (2) $-$ 2 * equation (1):

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 5 \end{array}\right]$$

$$\begin{array}{r} x_1 - x_2 = 5 \\ - \quad 2x_1 + 2x_2 = 8 \\ \hline -x_1 - 3x_2 = -3 \end{array}$$

$$x_1 - x_2 = 5$$
$$2x_1 + 2x_2 = (4)(2)$$

$\Rightarrow$ the equivalent system:

$$\cancel{x_1} + x_2 = 4 \quad --- (1)$$
$$-\cancel{x_1} - 3x_2 = -3 \quad --- (2)$$

$$-2x_2 = 1$$
$$\boxed{x_2 = \frac{-1}{2}}$$

$$-\cancel{x_1} - 3x_2 = -3$$
$$\cancel{x_1} - x_2 = 5$$
$$-4x_2 = 2$$
$$x_2 = \frac{-1}{2}$$

(1.2) Row Echelon Form

1) Row Echelon Form (REF).
2) Reduced Row Echelon Form (RREF).

main idea: transform any $(m \times n)$ system to REF or RREF using elementary row operations.

1) REF:

Def: A matrix M is said to be in Row Echelon Form, if:

1) The first non-zero entry in each non-zero row is one (this one called the leading one).
2) The leading ones in the lower rows is to the right of the above leading ones.
3) If there are any zero rows they must be at the bottom of the matrix.

ex: Are these matrices in REF?

1) $\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

solution: 1) No, because condition (1) didn't satisfy.
2) No, because condition (3) didn't satisfy.
3) No, because condition (2) didndn't satisfy.

2) RREF:

Def: A matrix M is said to be in Reduced Row Echelon Form, iff:

1) The matrix M is in Row Echelon Form.
2) The elements in the column of each leading one are all zeros.

ex: 1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2)

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1) Satisfies both REF and RREF.
2) Satisfies REF only.

ex1: Suppose that we transformed some system to the following REF.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 9 \end{array}\right] \Rightarrow \begin{array}{r} x_1 + 2x_2 + 3x_3 = 4 \quad ---(1) \\ x_2 + x_3 = 5 \quad ---(2) \\ x_3 = 9 \quad ---(3) \end{array}$$

using back substitution:

$\boxed{x_3 = 9}$ ---(3)

$x_2 + 9 = 5$ ---(2)

$\Rightarrow \boxed{x_2 = -4}$

$x_1 + 2(-4) + 3(9) = 4$ ---(3)

$x_1 - 8 + 27 = 4$

$\Rightarrow \boxed{x_1 = -15}$

solution set: $(9, -4, -15)$ (unique solution)

ex2: Given that the following matrix is the RREF of some system.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} x_1 = 2 \quad ---(1) \\ x_2 = -4 \quad ---(2) \end{array}$$

solution set: $(2, -4)$ (unique solution) ex3: The REF of some system is the following matrix

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \Longrightarrow \begin{array}{ll} x_2 + 2x_3 = 0 & ---(1) \\ x_3 = 4 & ---(2) \\ x_4 = 3 & ---(3) \\ 0 = 2 & ---(4) \end{array}$$

equation (4) is impossible.
$\Longrightarrow$ the system is inconsitent (has no solution)

important note: in REF and RREF, the variables corresponding to the leading ones are called the "leading variables". The other variables, if any, are called "free variables".

ex4: Suppose the following is the RREF for some system, find solution set.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 0 & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 \end{array}\right] \Longrightarrow \begin{array}{ll} x_2 + 2x_4 = 4 & ---(1) \\ x_3 + x_4 = 5 & ---(2) \\ x_5 = 6 & ---(3) \end{array}$$

leading variables: x_2, x_3, x_5.
free variables: x_1, x_4.

let $x_4 = s$, $x_1 = t$, $s, t \in \mathbb{R}$

$\Longrightarrow x_3 = 5 - s$, $x_2 = 4 - 2s$

$\Longrightarrow$ solution set: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t \\ 4-2s \\ 5-s \\ s \\ 6 \end{pmatrix}$, $t, s \in \mathbb{R}$

(the system has infinitely many solutions) note: if we have $(m \times n)$ system, the number of leading variables $\leq m$.

ex 5: solve this system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{array}\right] \Longrightarrow \begin{array}{ll} x_1 = 5 & ---(1) \\ x_2 + 2x_3 = 6 & ---(2) \\ 0 = 4 & ---(3) \end{array}$$

leading variables: x_1, x_2
free variables: x_3

from equation (3) the system is inconsistent.

How to transform a system to REF or RREF?

1) Gaussian Elemination: To transform a system to REF by elementary row operations.

2) Gauss Jordan Reduction: To transform a system to RREF by elementary row operations.

ex1: Solve the following system by Gaussian Elemination.

$$\begin{array}{ll} x_1 + 2x_2 + x_3 = 1 & ---(1) \\ 2x_1 + x_2 + x_3 = 2 & ---(2) \\ 4x_1 + 3x_2 + 3x_3 = 4 & ---(3) \\ 2x_1 + x_2 + 3x_3 = 5 & ---(4) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \end{array}\right] \longrightarrow \text{pivotal row}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right] \qquad \begin{matrix} (R_2 - 2R_1) \\ (R_3 - 4R_1) \\ (R_4 - 2R_1) \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right] \qquad (R_2 \times -\tfrac{1}{5}) \quad 560$$

560 2800 5 25 300

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right] \longrightarrow \begin{matrix} \text{pivotal row} \\ (R_3 + 5R_2) \\ (R_4 + 5R_2) \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left(\begin{matrix} \text{interchange} \\ R_3 \text{ with } R_4 \end{matrix}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (R_3 \times \tfrac{1}{2})$$

$$\Longrightarrow \text{REF}$$

leading variables: x_1, x_2, x_3
free variables: none

$x_1 + 2x_2 + x_3 = 1$ --- (1)
$x_2 + \frac{1}{5}x_3 = 0$ --- (2)
$x_3 = \frac{3}{2}$ --- (3)

$\Longrightarrow \boxed{x_3 = \frac{3}{2}}$

$x_2 + \frac{1}{5}x_3 = 0$ --- (2)
$x_2 + \frac{1}{5}\left(\frac{3}{2}\right) = 0$

$\Longrightarrow \boxed{x_2 = \frac{-3}{10}}$

$x_1 + 2x_2 + x_3 = 1$ --- (1)
$x_1 + 2\left(\frac{-3}{10}\right) + \frac{3}{2} = 1$

$\Longrightarrow \boxed{x_1 = \frac{1}{10}}$

$\Longrightarrow$ solution set: $\left(\frac{1}{10}, \frac{-3}{10}, \frac{3}{2}\right)$

ex2: Use Gauss Jordan Reduction to solve this system:

$-x_1 + x_2 - x_3 + 3x_4 = 0$ --- (1)

$3x_1 + x_2 - x_3 - x_4 = 0$ --- (2)

$2x_1 - x_2 - 2x_3 - x_4 = 0$ --- (3)

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ \text{[illegible]} & & & & \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right] \longrightarrow \text{pivotal row} \quad (R_1 \times -1)$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \quad \begin{array}{l} (R_2 - 3R_1) \\ (R_3 - 2R_1) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \quad (R_2 \times \tfrac{1}{4})$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \longrightarrow \begin{array}{l} \text{pivotal row} \\ (R_3 - R_2) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \quad (R_3 \times \tfrac{-1}{3})$$

$\Longrightarrow$ REF

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \\ \to \text{pivotal row} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} (R_1 - R_3) \\ (R_2 + R_3) \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} (R_1 + R_2) \\ \to \text{pivotal row} \\ \end{array}$$

$\Rightarrow$ RREF

$$\Rightarrow \begin{array}{ll} x_1 - x_4 = 0 & \text{---} (1) \\ x_2 + x_4 = 0 & \text{---} (2) \\ x_3 - x_4 = 0 & \text{---} (3) \end{array}$$

leading variables: x_1, x_2, x_3
free variables: x_4

let $x_4 = t$, $t \in \mathbb{R}$

$x_1 = x_4$ --- (1)
$\Rightarrow \boxed{x_1 = t}$
$x_2 = -x_4$ --- (2)
$\Rightarrow \boxed{x_2 = -t}$
$x_3 = x_4$ --- (3)
$\Rightarrow \boxed{x_3 = t}$

$\Rightarrow$ solution set: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \\ t \end{pmatrix}$

$t \in \mathbb{R}$

(this system has infinitely many solutions)

Hamogeneos System:

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = 0$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = 0$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = 0$$

This system is always conssistent because $x_1 = 0$, $x_2 = 0, \dots,$ $x_n = 0$ is a solution.

note: for an ~~underdetermined~~ under determined system $(m<n)$, there are only two cases:

1) The system is inconssitent. has no solution
2) The system has infinitely many solutions (because the number of ~~free variables~~ leading variables $\leq m$, so there must be some free variables.

result: a hamogeneos under determined system has always infinitely many solutions (because the hamogeneos system can't be inconsistent)

* Any solution ($X = 0$) of homogen
other than

leads to infinit many solutions.

excercise 10: Consider a linear system whose augmented matrix is of the form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

a) For what values of a and b will the system have infinitely many solutions?

b) For what values of a and b will the system be inconsistent?

c) For what values of a and b will the system have a unique solution?

solution: a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \rightarrow \text{pivotal row} \quad \begin{array}{l} (R_2 - R_1) \\ (R_3 - R_1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right] \rightarrow \text{pivotal row} \quad (R_3 - 2R_2)$$

$a=5$ and $b=4$ will make the final row a zero row, which leads to make x_3 a free variable and make the system has infinitely many solutions.

b) $a=5$ and $b \neq 4$ will make the system inconsistent.

c) $a-5 \neq 0 \Rightarrow a \neq 5$

$(b-4) \in \mathbb{R} \Rightarrow b \in \mathbb{R}$

will make the system has a unique solution.

(1.3) Matrix Arithmetic

Matrix: A rectangular array of m rows and n columns.

ex: A_{2*3} : A matrix of 2 rows and 3 columns.

* Size (order) of A_{m*n} is $m*n$.

* The entries of A_{m*n} are scalars.

* The (i,j) entry $= a_{ij}$

ith row   jth column

ex: $B = \begin{bmatrix} 1 & 2 & 3 & 8 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

size $= 2*4$
$b_{23} = 7$
$(1,2)$ entry $= b_{12} = 2$

note: $A_{m*n} = (a_{ij})$, $i = 1,2,3,\dots,m$
$j = 1,2,3,\dots,n$.

* The $n*1$ Matrix is called "column vector".

* The $1*n$ Matrix is called "row vector".

* Euclidean n-space $= R^n =$ All $n*1$ matrices with real entries.
$R^{1*n} =$ All $1*n$ matrices with real entries.
$R^{m*n} =$ All $m*n$ matrices with real entries.

ex1: $R^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} , a, b \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \in R^2$, $\begin{bmatrix} 1 \\ -4 \end{bmatrix} \in R^2$

ex2: $R^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , x_1, x_2, x_3 \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in R^3$

* Equality:

Two matrice A and B are equal, if:

1) size of A and B is the same,
2) $a_{ij} = b_{ij}$

ex: $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a \\ 4 & b-2 \\ 6 & 7 \end{bmatrix}$

if $A = B$, find a, b.

$\boxed{a = 2}$

$b - 2 = 5$

$\Longrightarrow \boxed{b = 7}$

* Scalar Multiplication:

if $A_{m \times n}$ is a matrix and α is a constant

$\alpha A = C$, such that:

$C_{ij} = \alpha a_{ij}$

* Matrix Adding:

if $A_{m \times n}$, $B_{m \times n}$ and $C_{m \times n}$, then:

$C = A + B$, such that:

$c_{ij} = a_{ij} + b_{ij}$

* Zero Matrix:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

* $A + O = O + A = A$

* $A - B = A + (-1 * B)$

* $A + {}^{-}A = O$

* $-A$ is the addative inverse of A.

$$* \; A_{m*n} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

we can deal with A_{m*n} as:

$$A = [a_1, a_2, a_3, \cdots, a_n] \;,\; a_j \in R^m \;,\; j = 1,2,3,\ldots,n$$

or $$A = \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix} \;,\; \vec{a_i} \in R^{1*n} \;,\; i = 1,2,3,\ldots,m$$

ex: $$A_{3*2} = \begin{bmatrix} 2 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\Longrightarrow a_1 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \in R^3 \;,\; a_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \in R^3$$

or $$\vec{a_1} = [2, 4] \in R^{1*2}$$
$$\vec{a_2} = [5, 6] \in R^{1*2}$$
$$\vec{a_3} = [7, 8] \in R^{1*2}$$

* Matrix Multiplication:

A_{m*n} , B_{n*p} , we define:

$$A_{m*n} * B_{n*p} = C_{m*p}$$

such that: $C_{ij} = \vec{a_{i..}} \cdot b_j = \sum_{K=1}^{n} a_{iK} \cdot b_{Kj}$

$$1*\boxed{n \quad n}*1 \Longrightarrow (1*1)$$

ex: $A = \begin{bmatrix} 2 & 4 \\ 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}$

$(2*2)$ $\qquad$ $(2*3)$

$$\Longrightarrow A * B = \begin{bmatrix} (2*2 + 4*6) & (2*4 + 4*7) & (2*5 + 4*7) \\ (6*2 + 4*6) & (6*4 + 4*7) & (6*5 + 4*7) \end{bmatrix}$$

$$A * B = \begin{bmatrix} 28 & 36 & 38 \\ 36 & 52 & 58 \end{bmatrix}$$

note: $B * A$ is not defined

$\Longrightarrow A * B \neq B * A$ (Matrix Multiplication is not Commutative)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$x \in R^n$, $b \in R^m$

This system is: $A * x = b$

ex1: $2x_1 + x_2 + x_3 = 0$
$x_1 - x_2 + x_3 = 4$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{(2*3)}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{(3*1)}, \quad b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}_{(2*1)}$$

So $A * x = b$

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{\begin{bmatrix} 2x_1 + x_2 + x_3 \\ x_1 - x_2 + x_3 \end{bmatrix}}$$

$$= \begin{bmatrix} 2x_1 + x_2 + x_3 \\ \dots \end{bmatrix} = b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

ex 2:

Ex 2: $2x_1 - x_2 = 1$ --- (1)

$x_1 + x_2 = 0$ --- (2)

$x_1 - 2x_1 = 4$ --- (3)

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \cancel{x_3} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix} = b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

Now: $\vec{a_1} = (2, -1)$ or $a_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

$\vec{a_2} = (1, 1)$

$\vec{a_3} = (1, -2)$

$$\Longrightarrow \vec{a_1} x = [2, -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1 - x_2$$

$$\vec{a_2} x = [1, 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2$$

$$\vec{a_3} x = [1, -2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 - 2x_2$$

$$b = Ax = \begin{bmatrix} \vec{a_1} & x \\ \vec{a_2} & x \\ \vec{a_3} & x \end{bmatrix}$$

in general, for (m×n) system, $Ax = b$:

$$b = Ax = \begin{bmatrix} \vec{a_1} & x \\ \vec{a_2} & x \\ \vdots & \vdots \\ \vec{a_m} & x \end{bmatrix}$$

Now: $b = Ax = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \\ -2x_2 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2$$

in general: for an (m×n) system $Ax = b$:

$$b = Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

* Linear Combination:

If v_1 v_2 v_k ... $\mathbb{R}^n$

A linear combination of these is:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

where: $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars.

ex1: $2\begin{pmatrix}1\\2\end{pmatrix} + 4\begin{pmatrix}0\\4\end{pmatrix}$ is a linear combination of the vectors:

$$v_1 = \begin{pmatrix}1\\2\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\4\end{pmatrix}$$

$$\alpha_1 = 2, \quad \alpha_2 = 4.$$

ex2: $\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} + 0\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

so we can say that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ can be written as

a linear combination of the vectors:

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}$$

Now, we Knew that ($m \times n$) system $Ax=b$:

$$b = Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

exs: $x_1 + x_2 = 5$
$2x_1 + 3x_2 = 13$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

This system has a solution (consistent) $x_1 = 2$, $x_2 = 3$ $\Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$b = Ax = x_1 a_1 + x_2 a_2$$

$$\overset{x_1}{2} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{a_1} + \overset{x_2}{3} \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{a_2} = \underbrace{\begin{pmatrix} 5 \\ 13 \end{pmatrix}}_{b} = b$$

So, b can be written as a linear combination of the column vectors of A.

result: If $Ax = b$ is consistent, then b can be written as a linear combination of the column vectors of A.

ex 4: $x_1 - x_2 - x_3 = -3$
$2x_1 + x_2 = 6$

$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $b = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$$\underbrace{\begin{pmatrix} -3 \\ 6 \end{pmatrix}}_{b} = \underbrace{2}_{x_1} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{a_1} + \underbrace{2}_{x_2} \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{a_2} + \underbrace{3}_{x_3} \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{a_3}$$

$\Rightarrow$ b can be written as a linear combination of a_1, a_2, a_3.

$\Rightarrow$ The system is conssistent.

solution: $(2, 2, 3)$

Th: $Ax = b$ is conssistent iff b can be written as a linear combination of the column vectors of A.

ex 5: Consider a (4×3) system $Ax = b$, $b = 2a_1$.
Is this system conssistent? If yes give one solution.

yes, $b = 2.a_1 + 0.a_2 + 0.a_3$ Linear Combination of a_1, a_2, a_3.

a solution: $(2, 0, 0)$

ex 6: consider a (4×4) system $A.x = 0$ and $b = 3a_2 - a_3$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a solution, since the system is homogeneos.

$\Rightarrow b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0.a_1 + 3a_2 - a_3 + 0.a_4$

$\Rightarrow \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix}$ is also a solution.

$\Rightarrow$ this system has infinitely many solutions, since it has more than one solution.

* Transpose:

Def: If A is a $(m*n)$ matrix, then the transpose of

ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 4 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

$(2,3)$ entry of $A^T = 4$
$(3,2)$ entry of $A = 4$

Def: an $(n*n)$ matrix A is called symmetric if $A^T = A$ ($a_{ji} = a_{ij}$ for all $i,j \in \{1,2,3,\dots\}$).

Def: An $(n*n)$ matrix A is called skew symmetric, if $A^T = -A$ ($a_{ji} = -a_{ij}$ for all $i,j \in \{1,2,3,\dots\}$).

ex1: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A = A^T \Longrightarrow A$ is symmetric.

ex2: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A^T = -A \Longrightarrow A$ is skew symmetric.

(1.4) Matrix Algebra

* Rules of Transposes

1) $(A^T)^T = A$

2) $(\alpha A)^T = \alpha A^T$

3) $(A+B)^T = A^T + B^T$

4) $(AB)^T = B^T A^T$

ex on point (4):

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 9 & 12 \\ 12 & 15 \end{bmatrix}$

$(BA)^T = \begin{bmatrix} 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$

$A^T B^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$

$\Longrightarrow (BA)^T = A^T B^T$

★ Rules:

1) $A + B = B + A$

2) $AB \neq BA$

3) $(A+B)+C = A + (B+C)$

4) $(A+B)C = AC + BC$

5) $C(A+B) = CA + CB$

6) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

7) $\alpha(A+B) = \alpha A + \alpha B$

8) $(\alpha + \beta)A = \alpha A + \beta A$

9) $(\alpha\beta)A = \alpha(\beta A) = \beta(\alpha A)$

10) $A^K = A * A \dots A$ (K times), $K \in \{1, 2, 3, \dots\}$

where A, B, C are matrices, α, β are scalars.

note: if $AB = 0$, then we cannot conclude if $A = 0$ or $B = 0$.

ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

* The Identity Matrix :

A matrix $(n \times n)$, such that $I = (\delta_{ij})$

$$\delta_{ij} = \begin{cases} 1 , & \text{if } i = j \\ 0 , & \text{if } i \neq j \end{cases}$$

i.e. the diagonal entries of I are all (1), the rest entries are (0).

ex1: I_2 : identity matrix of size (2×2) :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ex2: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark: If $A_{n \times n}$, then:

$$AI = IA = A$$

ex1: $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AI = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = A$$

ex2: A_{n*p} , B_{r*n} , I_n (identity matrix (n*n))

1) $A_{m\times p} * I_{n\times n}$ = not defined

2) $I_{n*n} * A_{n*p} = A_{n*p}$

3) $B_{r*n} * I_{n*n} = B_{r*n}$

4) $I_{n*n} * B_{r*n}$ = ~~B~~ not defined

* Properities of Identity Matrix:

1) $I^T = I$

2) Identity matrix always square matrix.

3) $I^k = I$, $k \in \{1, 2, 3, \ldots\}$

* Multiplicative Inverse

If A is (n*n) matrix , and there exists matrix B_{n*n} , such that:

$$AB = BA = I$$

Then we say that A is Invertible (non-singular) and $A^{-1} = B$.

note: If $A^{-1} = B$, then $B^{-1} = A$.

ex: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\Rightarrow A^{-1} = B$ and $B^{-1} = A$.

Th: if $A_{n \times n}$ has an Inverse, then this Inverse Is unique. (any square matrix has at most one inverse).

proof: Assume that B and C are two Inverses of A, then if we proved that $B = C$, then A has a unique Inverse.

B is the inverse of $A \Rightarrow AB = BA = I$
C is the Inverse of $A \Rightarrow AC = CA = I$

now: $CAB = C(AB) = CI = C$

also $CAB = (CA)B = IB = B$

$\Rightarrow B = C$

Th: If A, B are $(n \times n)$ non-singular matrices, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= (A\,I)A^{-1}$$
$$= A\,A^{-1}$$
$$= I$$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)\,B$

$$= (B^{-1}\,I)\cdot B$$
$$= B^{-1}\,B$$
$$= I$$

$\Longrightarrow$ $B^{-1}A^{-1}$ is the inverse of AB and vice versa.

remark: if $A_1, A_2, \dots, A_K$ are non-singular matrices, then

$A_1, A_2, \dots, A_K$ is non-singular and also

$$(A_1, A_2, \dots, A_K)^{-1} = A_K^{-1}, \dots, A_2^{-1}, A_1^{-1}$$

ex: $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

* Propercities of inverse:

1) $(A^{-1})^{-1} = A$

proof. $A^{-1}A = I$ and $AA^{-1} = I$

2) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$, α : scalar.

proof: $\alpha A \left(\frac{1}{\alpha} A^{-1}\right) = \frac{\cancel{\alpha}}{\cancel{\alpha}} AA^{-1} = I$

and $\frac{1}{\alpha} A^{-1}(\alpha A) = \frac{\cancel{\alpha}}{\cancel{\alpha}} A^{-1}A = I$

3) $(A^T)^{-1} = (A^{-1})^T$

proof: $A^T(A^{-1})^T = (A^{-1}A)^T$ $\quad ((AB)^T = B^T A^T)$

$= (I)^T = I$

and $(A^{-1})^T(A^T) = (AA^{-1})^T = (I)^T$

$= I$

4) $((AB)^T)^{-1} = (A^{-1})^T (B^{-1})^T$

proof: $(AB)^T(A^{-1})^T(B^{-1})^T = (B^T A^T)(A^T)^{-1}(B^T)^{-1}$

$= B^T (I) (B^T)^{-1}$

$= B^T (B^T)^{-1} = I$

or $((AB)^T)^{-1} = (B^T A^T)^{-1}$

$$= (A^T)^{-1} (B^T)^{-1}$$

$$= (A^{-1})^T (B^{-1})^T$$

(1.8) Elementary Matrices

We have three types of elementary matrices:

Type (I) , Type (II) , Type (III)

Def: An elementary matrix is a matrix obtained from the identity by doing exactly one elementary row operation.

Elementary Matrix Type (I) ($E^{(1)}$):

Obtained from doing row operation (I).

ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Longrightarrow$ $E^{(1)}$: Interchanging R_1 with R_2

$$E^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrix Type (II) ($E^{(2)}$):

Obtained from doing row operation (II).

ex 1: $E^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_1 * 2$

ex2: $E^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Longrightarrow I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_3 * -3$,

Elementary Matrix Type (III) ($E^{(3)}$):

Obtained from doing row operation (III).

ex 1: $E^{(3)} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \Longrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 \to R_2 - 2R_1$

ex 2: $E^{(3)} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_1 \to R_1 + \frac{1}{2}R_2$

ex1: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $E^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $R_1 \leftrightarrow R_3$

$$E^{(1)} A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}.$$

note: If $E^{(1)}$ is an elementary matrix of Type (I), then $(E^{(1)})^{-1}$ is also elementary matrix of Type (I) and $(E^{(1)})^{-1} = E^{(1)}$, because $E^{(1)} E^{(1)} = I$

ex 2: $E^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 * 4$ of I_3

$E^{(2)} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$E^{(2)} A = \begin{bmatrix} a & b & c \\ 4d & 4e & 4f \\ g & h & i \end{bmatrix}$, $R_2 * 4$ of A

note: $E^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{(2)} (E^{(2)})^{-1} = I$

$\Rightarrow (E^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 * \frac{1}{4}$ of I_3

ex3: $E^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 \to R_2 + 3R_1$ of I_3

$$E^{(3)} A = \begin{bmatrix} a & b & c \\ d+3a & e+3b & f+3c \\ g & h & i \end{bmatrix} , \quad R_2 \to R_2 + 3R_1 \text{ of } A$$

note: $\left(E^{(3)}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 \to R_2 - 3R_1$ of I_3 .

Def: let A, B be two matrices, then B is row equivalent to A, if there exists a finite number of elementary matrices of any type ~~$(E^{(1)}, E^{(2)}, E^{(3)})$~~, such that: $(E_1, E_2, \dots, E_K)$

$$B = E_K E_{K-1} \dots E_2 E_1 A$$

ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $E^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$B = E_2 E_1 A = E_2 (E_1 A) = E_2 \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 1 & 2 \end{bmatrix}$$

$\Rightarrow$ B is row equivalent to A.

Th: If A is $(m \times n)$ matrix, then the following statements are equivalent:

a) A is non-singular.
b) $A\underline{x} = b$ has a unique solution for any $b \in \mathbb{R}^n$.
c) $A\underline{x} = 0$ has only the trivial solution.
d) A is row equivalent to I (the RREF of A is I).

proof: $a \to b \to c \to d \to a$ (circle)

1) $a \to b$: Assume that A is non-singular.
(we want to show that $A\underline{x} = b$ has a unique solution.)

A is non-singular $\Rightarrow$ A^{-1} exists

so $A\underline{x} = b$

$$A^{-1} A \underline{x} = A^{-1} b$$
$$I \underline{x} = A^{-1} b$$
$$\underline{x} = A^{-1} b$$

so $\underline{x} = A^{-1} b$ is the only solution of $A\underline{x} = b$.

2) $b \to c$: Assume that $A\underline{x} = b$ has a unique solution for any $b \in \mathbb{R}^n$. (we want to show that $A\underline{x} = 0$ has only the trivial solution).

Take $b = 0$ $\Rightarrow$ $A\underline{x} = 0 = b$ has a unique solution which is the trivial solution.

3) $c \to d$: Assume that $Ax = 0$ has only the trivial solution

Since $A\alpha = 0$ has only one solution which is the trivial solution, the augmented matrix:

$$(A \mid 0) = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

$\Longrightarrow$ this matrix only contains the leading ones and some zeros.

$\Longrightarrow$ A is in RREF.

$\Longrightarrow$ A is row equivalent to I.

4) $d \to a$: Assume A is row equivalent to I, then:

$$A = E_K E_{K-1} \cdots E_2 E_1 I_n$$
$$A = E_K E_{K-1} \cdots E_2 E_1$$

since each elementary matrix is non-singular & the product of non-singular matrices is also non-singular, then we have A is non-singular.

Result: A is non-singular, iff the system $A\alpha = b$ has a unique solution.

note: A is non-singular, iff $x = 0$ is the only solution for the system $A\alpha = 0$. (A is singular, iff $Ax = 0$ has a non-zero solution)

How to find the inverse of A?

Apply Gauss – Jordan Reduction to $(A \mid I)$.

The result will be $(C \mid D)$.

If $C = I$, then $D = A^{-1}$

If $C \neq I$, then A is singular.

ex: $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ Find A^{-1} if possible.

solution: $\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array}\right] \longrightarrow$ pivotal row $(R_2 - R_1)$ $(R_3 - R_1)$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \longrightarrow$ pivotal row $(R_3 - 2R_2)$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -11 & 8 & -4 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} (R_1 - 4R_3) \\ (R_2 - 2R_3) \\ \longrightarrow \text{pivotal row} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} (R_1 - 2R_2) \\ \longrightarrow \text{pivotal row} \\ \end{array}$$

$$C = I \implies A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

How to use the inverse to solve some square systems?

ex: Consider a (2*2) system:

$$2x_1 + x_2 = 1$$
$$3x_1 + 2x_2 = 4$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

a) find A^{-1} if possible

$(A | I)$

$$= \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \quad (R_1 * \tfrac{1}{2})$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{-3}{2} & 1 \end{array}\right] \rightarrow \text{pivotal row} \quad (R_2 - 3R_1)$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 2 \end{array}\right] \quad (R_2 * 2)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right] \quad (R_1 - \tfrac{1}{2}R_2) \rightarrow \text{pivotal row}$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} 2 & -1 \\ -3 & 2 \end{matrix}}_{A^{-1}}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

b) find the unique solution.

$$Ax = b$$

$$x = A \quad b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

LU Factorization:

Def: An $(n \times n)$ matrix A is called "upper triangular", if the entries below the main diagonal are all zeros (i.e. $a_{ij} = 0$, when $i > j$)

ex:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \Longrightarrow \text{upper triangular}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \text{upper triangular}$$

Def: An $(n \times n)$ matrix A is called "lower triangular", if the entries above the main diagonal are all zeros (i.e. $a_{ij} = 0$, when $i < j$)

ex:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Longrightarrow \text{lower triangular}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \Longrightarrow \text{lower triangular}$$

Def: An $(n \times n)$ matrix is called "triangular", if it is either upper triangular or lower triangular.

ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ ~~triangular~~ diagonal
(this matrix is upper triangular and lower triangular which called "diagonal")

LU Factorization: write a matrix $A_{n \times n}$ as a product of two matrices L & U, such that : $A = LU$, where L: lower Δ, U: upper Δ.

How to do so?

Answer: We use <u>only</u> row operation (III).

ex: $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$, Find LU factorization for A.

solution: $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$

$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$ $(R_2 - \frac{1}{2} R_1) \Rightarrow l_{21} = \frac{1}{2}$
$(R_3 - 2R_1) \Rightarrow l_{31} = 2$

$$\cancel{tt}\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = u$$

$$(R_3 + 3R_2) \Longrightarrow l_{32} = -3$$

now: $l_{21} = \frac{1}{2}$, $l_{31} = 2$, $l_{32} = -3$

$$\Longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\Longrightarrow A = Lu = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

ex2: $A = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$, Find Lu factorization for A.

solution: $\begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 0 & \frac{10}{3} \end{bmatrix} = u$$

$$(R_2 - \frac{4}{3} R_1) \Longrightarrow l_{21} = \frac{4}{3}$$

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{bmatrix}$$

$$A = Lu = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & \frac{10}{3} \end{bmatrix}$$

Where did L come from?

Answer: take ex2 for example:

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} \xrightarrow{R_2 - \frac{4}{3}R_1} Lu = \begin{bmatrix} 3 & 2 \\ 0 & \frac{10}{3} \end{bmatrix}$$

$$\Longrightarrow I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{4}{3}R_1} E^{(3)} = \begin{bmatrix} 1 & 0 \\ -\frac{4}{3} & 1 \end{bmatrix}$$

$$\text{So, } EA = \begin{bmatrix} 1 & 0 \\ \frac{-4}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & \frac{10}{3} \end{bmatrix} = u$$

now, $EA = u$

$$E^{-1}(EA) = E^{-1}u$$

$$A = E^{-1}u \quad , \text{ but } A = Lu$$

$$\Longrightarrow \boxed{L = E^{-1}}$$

Notes:

1) If A is row equivalent to B, then B is row

$$A = E_K E_{K-1} \cdots E_2 E_1 B$$

$$B = (E_K E_{K-1} \cdots E_2 E_1)^{-1} A$$
$$= E_1^{-1} E_2^{-1} \cdots E_{K-1}^{-1} E_K^{-1} A$$

2) Standard Vectors in $\mathbb{R}^n$

ex 1: $\mathbb{R}^2$: $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with columns e_1, e_2

$\Rightarrow e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

e_1, e_2 are the standard vectors in $\mathbb{R}^2$.

ex 2: $\mathbb{R}^3$: $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with columns e_1, e_2, e_3

e_1, e_2, e_3 are the standard vectors in $\mathbb{R}^3$.

3) $A_{m\times n}$, $B_{n\times r} = (b_1 \; b_2 \; \cdots \; b_r)$

$$AB = A\,(b_1 \; b_2 \; \cdots \; b_r)$$

$$AB = (Ab_1 \; Ab_2 \; \cdots \; Ab_r)$$

ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{array}{ccc} \left[\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{matrix}\right] \\ \begin{matrix} b_1 & b_2 & b_3 \end{matrix} \end{array}$

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

$$\Longrightarrow AB = (Ab_1 \; Ab_2 \; Ab_3)$$

$$AB = \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 4 \end{bmatrix}$$

also: $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 4 \end{bmatrix}$$

(1.5 − 20) If $A_{n\times n}$ is non-singular, $B_{n\times r}$; then show that the RREF of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

proof: A is non-singular $\Longrightarrow$ A is row equivalent to I.

$$E_K E_{K-1} \cdots E_2 E_1 A = I$$

$$\Longrightarrow E_K E_{K-1} \cdots E_2 E_1 = A^{-1}$$

So, $(E_K E_{K-1} \cdots E_2 E_1) B = A^{-1}B = C$

$\Longrightarrow$ C is row equivalent to $B \Rightarrow (I|C)$ is the RREF of $(A|B)$.

(1.5 − 18) let A, B be $(n\times n)$ matrices and $C = AB$, prove that if B is singular, then C must be singular.

method 1:
(using contradiction)

Suppose C is non-singular $\Longrightarrow C^{-1}$ exists.

Now, $C = AB$

$\Longrightarrow C^{-1} = (AB)^{-1}$

$C^{-1} = B^{-1}A^{-1}$

which mean B^{-1} exists
and so B is singular.
$\Longrightarrow$ Contradiction
$\Longrightarrow$ C is singular.

method 2:
using Th 1.5.2) $Cx = 0$ must have a non-zero solution iff C is singular.

Now, B is singular $\Rightarrow$ there is a nonzero solution for $B\alpha = 0$.

$$B\alpha = 0$$
$$A(B\alpha) = A(0)$$
$$(AB)\alpha = 0$$
$$C\alpha = 0$$

But $\alpha \neq 0 \Rightarrow$ C is singular.