Ch. 6

(6.1) Eigenvalues & Eigen Vectors

Def: Let A be an $(n * n)$ matrix, then a scalar λ is called an eigen value for A, if there is a [illegible]

$$A x = \lambda x$$

The vector x we found is called an eigen vector corresponding to λ.

note: eigen value is also called characteristic value, and eigen vector is also called characteristic vector.

note: eigen value $\equiv$ eigen wert.

---

ex: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$A x = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$A x = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 x$$

$\Rightarrow$ $\lambda = 5$ is the eigen value of A, and

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 5$.

note: if λ is an eigen value for A with corresponding eigen vector x, then any non-zero multiple of x is also an eigen value corresponding to: λ..

proof: $Ax = \lambda x$, we need to show that αx; $\alpha \neq 0$ is also an eigen vector corresponding to λ, that is we need to show that $A(\alpha x) = \lambda(\alpha x)$

$$A(\alpha x) = \alpha(Ax)$$
$$= \alpha(\lambda x)$$
$$= \lambda(\alpha x)$$

ex: take $\alpha = -2$, $\alpha x = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$A(\alpha x) = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ -10 \end{bmatrix} = 5\begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$= 5x$$

$\Rightarrow \lambda = 5.$

Q: How to find the eigenvalue of the corresponding eigenvalue?

$$Ax = \lambda I x \qquad , I: n*n \text{ identity matrix}$$

$$Ax - \lambda I x = 0$$

$$\underbrace{(A - \lambda I)}_{n*n \text{ matrix}} x = 0 \qquad (\text{homogeneous system}) \quad (x \neq 0)$$

Th: If A_{n*n} matrix and λ ~~scaller~~ scalar, then the following are equivalent:

a) λ is an eigenvalue for A.
b) $(A - \lambda I)x = 0$ has a non-trivial solution.
c) $N(A - \lambda I) \neq \{0\}$
d) $A - \lambda I$ is singular.
e) $\det(A - \lambda I) = 0$.

notes: 1) $N(A - \lambda I)$, the null space of the matrix $A - \lambda I$ is a subspace of $\mathbb{R}^n$ and called the eigenspace corresponding to λ.
Eigenspace corresponding to λ consists of all eigenvectors of λ in addition to the zero-vector.

2) $\det(A - \lambda I) = P(\lambda)$ is called the characteristic polynomial.

3) $\det(A - \lambda I) = 0$ is called the charateristic equation.

* the roots (zeros) of $P(\lambda)$ are the eigen-values of A.

ex: Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, Find the following:

1) Find the eigenvalues of A.
2) ~~Find~~ For each eigenvalue, find the corresponding eigenspace and its basis and its dimension.

1) $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$
$$3 - 4\lambda + \lambda^2 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda + 1)(\lambda - 5) = 0$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 5$$

2) For $\lambda_1 = -1$

eigenspace $= N(A+I)$

$$A+I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2t \quad , \quad t \in \mathbb{R}$

So, the eigenspace of $\lambda_1 = -1$ is

$$\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix} ; t \in \mathbb{R} \right\}$$

Basis for eigenspace of λ_1 is:

$$\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix} ; t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -2 \\ 1 \end{pmatrix} ; t \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

So, the basis is: $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim = 1$

Now, for $\lambda_2 = 5$

eigen space $= N(A-5I)$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2 = w$

So, eigen space of λ_2 is

$$\left\{ \begin{pmatrix} w \\ w \end{pmatrix} ; w \in \mathbb{R} \right\} = \left\{ w \begin{pmatrix} 1 \\ 1 \end{pmatrix} ; w \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$\therefore$ So, a basis is $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \dim = 1$

note: for the previous example:

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\lambda_1 = -1$, $\lambda_2 = 5 \Rightarrow$ 1) $\lambda_1 + \lambda_2 = a_{11} + a_{22}$

2) $\lambda_1 \lambda_2 = |A|$

Def: If $A_{n \times n}$ matrix, the trace of A denoted by $tr(A)$ is the sum of the diagonal elements of A.

i.e. $tr(A) = a_{11} + a_{22} + \cdots = \sum_{i=1}^{n} a_{ii}$

* In general, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalue of $A_{n \times n}$ (counting multiplicity), then:

1) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = tr(A)$

2) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = |A|$

Th: $A_{n\times n}$ is non-singular, iff $\lambda = 0$ is not an eigenvalue of A.

Def: Similar Matrices,

If A, B are $n\times n$ matrices, then we say that B is similar to A, if there is a non-singular matrix X, such that $B = X^{-1}AX$.

Th: Similar Matrices have the same characteristic polynomial and therefore, the same eigenvalues.

ex: let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} &(2-\lambda)\big((-\lambda-2)(+2-\lambda)\big) \\ &- \big(-3(2-\lambda)+3\big) \\ &+ \big(-3-(-2-\lambda)\big) \end{aligned} = 0$$

$$\Rightarrow \quad -\lambda\,(\lambda-1)\,(\lambda-1) = 0$$

$$\lambda_1 = 0 \;,\; \lambda_2 = 1 \;,\; \lambda_3 = 1$$

So, we have two distinct eigenvalues:

For $\lambda_1 = 0$:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

let $x_3 = t$, $t \in \mathbb{R}$, then:

$x_1 = t$, $x_2 = t$

So, the eigen space corresponding to $\lambda_1 = 0$:

$$\left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} \right\} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; t \in \mathbb{R} \right\}$$

Basis : $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, dim $= 1$

For $\lambda_2 = 1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_2 = t$, $x_3 = w$, $t, w \in \mathbb{R}$

$x_1 = 3t - w$

So, the eigenspace corresponding to $\lambda_2 = 1$:

$$\left\{ \begin{pmatrix} 3t - w \\ t \\ w \end{pmatrix} \right\} = \left\{ t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; t, w \in \mathbb{R} \right\}$$

Basis : $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, dim $= 2$

note : For the previous example :

$$\lambda_1 = 0 \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = 1 \longrightarrow \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

the 3 vectors are L.I .

(6.3) Diagonalization

Th: If A is $n \times n$ matrix, and $\lambda_1, \lambda_2, \dots, \lambda_K$ are distinct eigenvalues of A with corresponding eigenvectors $x_1, x_2, \dots, x_K$, then $x_1, x_2, \dots, x_K$ are L.I.

Def: Let A be a $(n \times n)$ matrix, we say that A is diagonalizable, if there exist a non-singular matrix X and a diagonal matrix D, such that:

$$X^{-1} A X = D \implies A = X D X^{-1}$$

notes: 1) A is diagonalizable, iff A is similar to a diagonal matrix D.

2) We say X diagonalize A.

Th: $A_{n \times n}$ is diagonalizable, iff A has n linearly independent ~~eigenvalues~~ eigenvectors.

$$\begin{array}{cccc} (x_1, & x_2, & \dots, & x_n) \\ \downarrow & \downarrow & & \downarrow \\ (\lambda_1, & \lambda_2, & \dots, & \lambda_n) \end{array}$$

the eigenvalues are possible to be repeated.

(6.3) Diagonalization

Th: If A is $n \times n$ matrix, and $\lambda_1, \lambda_2, \dots, \lambda_K$ are distinct eigenvalues of A with corresponding eigenvectors $x_1, x_2, \dots, x_K$, then $x_1, x_2, \dots, x_K$ are L.I.

Def: Let A be a $(n \times n)$ matrix, we say that A is diagonalizable, if there exist a non-singular matrix X and a diagonal matrix D, such that:

$$X^{-1} A X = D \Longrightarrow A = X D X^{-1}$$

notes: 1) A is diagonalizable, iff A is similar to a diagonal matrix D.

2) We say X diagonalize A.

Th: $A_{n \times n}$ is diagonalizable, iff A has n linearly independent ~~eigenvalues~~ eigenvectors.

$$\begin{array}{cccc} (x_1, & x_2, & \dots, & x_n) \\ \downarrow & \downarrow & & \downarrow \\ (\lambda_1, & \lambda_2, & \dots, & \lambda_n) \end{array}$$

the eigenvalues are possible to be repeated.

For $\lambda_2 = 1$

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_2 = t$, $x_3 = w$, $t, w \in \mathbb{R}$

$x_1 = 3t - w$

So, the eigenspace corresponding to $\lambda_2 = 1$:

$$\left\{ \begin{pmatrix} 3t - w \\ t \\ w \end{pmatrix} \right\} = \left\{ t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; t, w \in \mathbb{R} \right\}$$

Basis : $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, dim $= 2$

note : For the previous example :

$$\lambda_1 = 0 \qquad\qquad \lambda_2 = 1$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The 3 vectors are L.I .

note : $A_{n \times n}$ matrix , then :

1) If A has n distinct eigenvalues A is diagonalizable.

2) If A has less than n distinct eigenvalues, then A may or may not be diagonalizable.

3) Diagonalization is not unique.

* Application:

Finding powers of A :

$$A = X D X^{-1}$$

$$A^2 = (X D X^{-1})(X D X^{-1}) = X D^2 X^{-1}$$

$$A^3 = X D^3 X^{-1}$$

$$\vdots \qquad \vdots$$

$$A^K = X D^K X^{-1}$$

where $D^K = \begin{bmatrix} (d_{11})^K & 0 & 0 \\ 0 & (d_{22})^K & \vdots \\ 0 & 0 \cdots & (d_{nn})^K \end{bmatrix}$

$\Longrightarrow \quad X = (x_1 \quad x_2 \quad \dots \quad x_n)$ non-singular

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & & & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

ex: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

a) Find eigenvalues with corresponding eigenspace

b) Is A diagonalizable? If yes, find X, D such that $X^{-1}AX = D$.

a) $\lambda = -1$, $\lambda_2 = 5$ (see ex1 in section 6.1)

$$x_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $X = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$

$$X^{-1}AX = D$$

For λ_1:

$$\begin{bmatrix} 2i & 4 \\ -1 & 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + {-2i}\, x_2 = 0 \quad \Rightarrow \quad \boxed{x_1 = 2i\, x_2}$$

So, eigenspace of λ_1:

$$\left\{ \begin{pmatrix} 2i\,t \\ t \end{pmatrix} ; \ t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 2i \\ 1 \end{pmatrix} ; \ t \in \mathbb{R} \right\}$$

A basis is $\left\{ \begin{pmatrix} 2i \\ 1 \end{pmatrix} \right\}$

for λ_2:

The eigenspace will be $\left\{ w \begin{pmatrix} -2i \\ 1 \end{pmatrix} , \ w \in \mathbb{R} \right\}$

And the basis will be $\left\{ \begin{pmatrix} -2i \\ 1 \end{pmatrix} \right\}$.

ex1: $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

$\lambda_1 = 0 \ , \ \lambda_2 = 1$ (see the last ex in sec

$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} , \ x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ A is diagonalizable.

$X = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

---

ex 2: $A = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = -4$

$1-\lambda = \mp\sqrt{-4}$

$1-\lambda = \mp 2i$

$\Rightarrow \lambda_1 = 1-2i \ , \ \lambda_2 = 1+2i$