Q1: Show that $|AB| = |A||B|$.

proof: Take two cases:

case 1: B is singular $\Longrightarrow$ AB is singular.

B is singular $\Longrightarrow |B| = 0 \Longrightarrow |A|\,|B| = 0$
AB is singular $\Longrightarrow |AB| = 0$

So, $|AB| = |A||B|$

case 2: B is non-singular $\Longrightarrow$ B is row equivalent to I

$\Longrightarrow B = E_K E_{K-1} \dots E_2 E_1 I$
$\Longrightarrow B = E_K E_{K-1} \dots E_2 E_1$

$AB = A\, E_K E_{K-1} \dots E_2 E_1$

$|AB| = |A|\, |E_K E_{K-1} \dots E_2 E_1|$

$\Longrightarrow |AB| = |A|\,|B|$

note: Any non-singular matrix can be written as a product of elementary matrices.

Q2: Show that $A_{n \times n}$ is singular, iff $|A| = 0$.

$R = E_K E_{K-1} \dots E_2 E_1 A$ (where R is the REF of A)

$|R| = |E_K| |E_{K-1}| \dots |E_2| |E_1| |A|$

But, $|E_i| \neq 0$, for all i.

So, $|R| = 0$, iff $|A| = 0$

Now, A is singular, iff R contains a row consisting entirely of zeros.

And, R contains a row consisting entirely of zeros, iff $|R| = 0$.

But, $|R| = 0$, iff $|A| = 0$.

So, $A_{n \times n}$ is singular, iff $|A| = 0$.

Chapter 3: Vector Spaces

(3.1) Definitions & Examples

Def: A vector space V is a set of elements (elements called vectors) with [illegible]

addition and scalar multiplication, such that the following ten conditions are satisfied:

(C1) 1) $u + v \in V$, for any $u, v \in V$.

(C2) 2) $\alpha u \in V$, for $u \in V$, $\alpha \in \mathbb{R}$.

(A1) 3) $u + v = v + u$, for any $u, v \in V$.

(A2) 4) $u + (v + w) = (u + v) + w$, for any $u, v, w \in V$.

(A3) 5) There is an element in V called the zero vector denoted by 0, such that $u + 0 = 0 + u = u$, for any $u \in V$.

(A4) 6) For any $u \in V$, there is an element $-u$, such that $u + (-u) = 0$.

(A5) 7) $\alpha(u + v) = \alpha u + \alpha v$, for any $u, v \in V$, $\alpha \in \mathbb{R}$.

(A6) 8) $(\alpha + \beta) u = \alpha u + \beta u$, for any $u \in V$, $\alpha, \beta \in \mathbb{R}$.

(A7) 9) $(\alpha \beta) u = \alpha(\beta u) = \beta(\alpha u)$, for any $u \in V$, $\alpha, \beta \in \mathbb{R}$

(A8) 10) $1 * u = u$, for any $u \in V$, $1 \in \mathbb{R}$.

ex1: Let $V = \mathbb{R}^2$ with usual $(+)$ and $(\cdot)$, show that V is a vector space:

let $u = \begin{pmatrix} a \\ b \end{pmatrix}$, $v = \begin{pmatrix} c \\ d \end{pmatrix}$, $w = \begin{pmatrix} e \\ f \end{pmatrix}$

1) $u + v = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \in V$

2) $\alpha V = \alpha v = \begin{pmatrix} \alpha c \\ \alpha d \end{pmatrix} \in V$

3) $u+v = \begin{pmatrix} a+e \\ b+d \end{pmatrix}$, $v+u = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$

$\Rightarrow u+v = v+u$

4) $u+(v+w) = \begin{pmatrix} a \\ b \end{pmatrix} + \left[\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \right]$

$= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c+e \\ d+f \end{pmatrix}$

$= \cancel{+} \begin{pmatrix} a+b+e \\ b+d\ f \end{pmatrix}$

$(u+v)+w = \left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \right] + \begin{pmatrix} e \\ f \end{pmatrix}$

$= \begin{pmatrix} a+c \\ b+d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$

$= \begin{pmatrix} a+c+e \\ b+d+f \end{pmatrix}$

$\Rightarrow u+(v+w) = (u+v)+w$

5) $V + 0 = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

$0 + V = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\Longrightarrow V + 0 = 0 + V = V$, 0 is zero vector.

6) $V = \begin{pmatrix} c \\ d \end{pmatrix}$, $-V \equiv \begin{pmatrix} -c \\ -d \end{pmatrix}$

$V + (-V) = 0$

7) $\alpha(u+v) = \alpha \begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} \alpha a + \alpha c \\ \alpha b + \alpha d \end{pmatrix}$

$\alpha u = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$, $\alpha V = \begin{pmatrix} \alpha c \\ \alpha d \end{pmatrix} \Longrightarrow \alpha u + \alpha V = \begin{pmatrix} \alpha a + \alpha \\ \alpha b + \alpha \end{pmatrix}$

$\Longrightarrow \alpha(u+v) = \alpha u + \alpha V$

8) $(\alpha \beta) V = \begin{pmatrix} \alpha \beta c \\ \alpha \beta d \end{pmatrix}$, $\alpha(\beta V) = \alpha \begin{pmatrix} \beta c \\ \beta d \end{pmatrix} = \begin{pmatrix} \alpha \beta c \\ \alpha \beta d \end{pmatrix}$

$\Longrightarrow (\alpha \beta) V = \alpha(\beta V)$

9) $(\alpha + \beta) V = \begin{pmatrix} (\alpha+\beta) c \\ (\alpha+\beta) d \end{pmatrix}$, $\alpha V = \begin{pmatrix} \alpha c \\ \alpha d \end{pmatrix}$, $\beta V = \begin{pmatrix} \beta c \\ \beta d \end{pmatrix}$

$\Longrightarrow (\alpha+\beta) V = \alpha V + \beta V$

10) $1 * V = 1 * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = V$

ex2: $\mathbb{R}^n$ is a vector space.

proof: imitate the previous example.

ex3: $\mathbb{R}^{m\times n}$ is all matrices with real entries, and it is a vector space.

take for example $\mathbb{R}^{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $a_{ij} \in \mathbb{R}$

ex4: P_n is all polynomials of degree less than n,

$$P_n = \{ a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x^1 + a_0 \,,\, a_i \in \mathbb{R} \}$$

with usual $(+)$ and $(\cdot)$, such that:

$$(f+g)(x) = f(x) + g(x) \,,\, (\alpha f)(x) = \alpha f(x)$$

$\Longrightarrow P_n$ is a vector space.

proof: C_1: $f(x)$ & $g(x)$ are of degree less than n, so $(f+g)(x)$ is a polynomial of degree less than n.

ex: $f(x) = x^2 + 2x + 4$, $g(x) = 2x + 3$, $f, g \in P_3$

$\Longrightarrow (f+g)(x) = x^2 + 4x + 7 \in P_3$

C_2: $(\alpha f)(x) = \alpha f(x) \in P_n$, $f \in P_n$

$A_1, A_2, (A_5 - A_8)$ are satisfied.

A_3: $0 = 0 \cdot x^{n-1} + 0 \cdot x^{n-2} + \dots + 0$

A_4: $-f(x) = (-f)(x)$

ex5: $C[a,b]$ is all real valued functions (their range is $\mathbb{R}$) ~~are~~ that are continuous on $[a,b]$.

$f(x), g(x) \in C[a,b]$

1) $(f+g)(x) = f(x) + g(x) \in C[a,b]$

2) $(\alpha f)(x) = \alpha f(x) \in C[a,b]$

and so on for the other eight conditions.

ex6: Let $w = \left\{ \begin{pmatrix} 1 \\ a \end{pmatrix}, a \in \mathbb{R} \right\}$ with usual $(+)$ and $(\cdot)$.

Is w a vector space?

No, take $u = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $u, v \in w$

But, $u + v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \notin w$.

ex7: Let $u = \mathbb{R}^2$ with usual $(+)$, but:

$\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ x_2 \end{pmatrix}$, is u a vector space?

$(\alpha + B) u = \alpha u + B u$?

$(\alpha + B) u = \begin{pmatrix} (\alpha + B) x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + B x_1 \\ x_2 \end{pmatrix}$

$$\alpha u + \beta u = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 \\ \cancel{\alpha} x_2 \end{pmatrix} + \begin{pmatrix} \beta x_1 \\ \cancel{\beta} x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 + \beta x_1 \\ \cancel{\alpha} x_2 + \cancel{\beta} x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta x_1 \\ 2x_2 \end{pmatrix}$$

So, $(\alpha + \beta) u \neq \alpha u + \beta u$

$\Longrightarrow$ It isn't a vector space.

Th: If V is a vector space and $u \in V$, then:

1) $0 \cdot u = 0$, 0 is the zero vector.
2) $u + v = 0 \Longrightarrow v = -u$.
3) $(-1) u = -u$.

proof: 1) $0 = u + (-u)$

$$= 1 \cdot u + (-u)$$

$$= (1+0) u + (-u)$$

$$= 1u + 0 \cdot u + (-u)$$

$$= u + 0 \cdot u + (-u)$$

$$\Longrightarrow 0 = 0 \cdot u$$

2) $u + v = 0$

$$(-u) + (u+v) = 0 + (-u)$$

$$(-u + u) + v = 0 + (-u)$$

$$0 + v = (-u)$$

$$\Longrightarrow v = (-u)$$

3) $(-1)u = -u$:

$$0 = 0 \cdot u$$

$$= (1 + (-1))\, u$$

$$= 1 \cdot u + (-1)u$$

$$0 = u + (-1)u$$

$$\Longrightarrow (-1)u = -u$$

(let $v = (-1)u$ and use the previous rule)

Q: If V is a vector space and $u+v = u+w$, then $v = w$, prove that.

proof: $u+v = u+w$

$$((-u)+u)+v = ((-u)+u)+w$$

$$0 + v = 0 + w$$

$$\Longrightarrow v = w$$

ex: V is all ordered pairs in $\mathbb{R}$.

$$V = \{ (x,y),\ x \in \mathbb{R},\ y \in \mathbb{R} \}$$

under usual addition $(+)$: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
and usual scalar multiplication: $\alpha (x,y) = (\alpha x, \alpha y)$

So, V is a vector space.

note: in the previous example:

1) $0 = (0,0)$

2) $-u = (-x,-y)$, $u \in V$

ex 2: $T = \{ (x,y) \mid x, y \in \mathbb{R} \}$

$(+)$: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(\cdot)$: $\alpha (x,y) = (\alpha x, y)$

Is T a vector space?

No; because A_6 doesn't satisfied

A_6: $(\alpha + \beta) u = \alpha u + \beta u$, $u \in T$

take $\alpha = 1$, $\beta = 2$, $u = (-1,3)$

$$
\begin{aligned}
(\alpha + \beta) u &= (1+2)(-1,3) \\
&= 3(-1,3) \\
&= (-3,3)
\end{aligned}
$$

$$
\begin{aligned}
\alpha u + \beta u &= 1(-1,3) + 2(-1,3) \\
&= (-1,3) + (-2,3) \\
&= (-3,6)
\end{aligned}
$$

So, $(\alpha + \beta) u \neq \alpha u + \beta u$

$\Longrightarrow$ T is not a vector space ex3: $W = \{ (x,y) \mid x, y \in \mathbb{R} \}$

$(+)$: $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 0)$
$(\cdot)$: $\alpha (x,y) = (\alpha x, \alpha y)$

Is W a vector space?

No, since A_3 doesn't satisfied.

A_3: there is a 0, such that $u + 0 = u$.

take $u = (1,2)$, $0 = (0,0)$, $u \in W$

$u + 0 = (1,2) \oplus (0,0) = (1,0)$

So, $u + 0 \neq u$

$\Longrightarrow$ W is not a vector space.

ex4: $W = \mathbb{R}^+$ (positive numbers)

$(+)$: $a + b = a + b$ (usual addition)
$(\cdot)$: $\alpha a = \alpha a$ (usual scalar multiplication)

Is W a vector space?

No, since C_2 does not satisfy

take $\alpha = -1$, then $\alpha a = -a \notin \mathbb{R}^+$

So, W is not a vector space.

ex5: $W = \mathbb{R}^+$

$(+)$: $x \oplus y = x \cdot y$

$(\cdot)$: $\alpha \circ y = y^\alpha$

Is W a vector space?

C_1 & C_2 are satisfied, since $W = \mathbb{R}^+$.

A_1: $x \oplus y = x \cdot y$
$y \oplus x = y \cdot x$] equal

A_2: $x \oplus (y \oplus z) = x \cdot (y \cdot z) = xyz$
$(x \oplus y) \oplus z = (x \cdot y) \cdot z = xyz$] equal

A_3: There is a zero vector 0, such that $x \oplus 0 = x$
, which is 1, since $x \oplus 1 = x(1) = x$

A_4: $u \oplus (-u) = u \oplus u^{-1} = u \cdot u^{-1} = 1 = 0$

A_5: $\alpha \circ (x \oplus y) = (x \oplus y)^\alpha$
$= (x \cdot y)^\alpha$
$= x^\alpha \cdot y^\alpha$
$= (\alpha \circ x) \oplus (\alpha \circ y)$

A_6: $(\alpha + \beta) \circ x = x^{(\alpha + \beta)}$
$= x^\alpha \cdot x^\beta$
$= (\alpha \circ x) \oplus (\beta \circ x)$

(3.2) Subspaces

Def: Let V be a vector space and S a subset of V, then S is called a subspace of V, if setself is a vector space under (+) and (·) of V.

Th: If V is a vector space and S a subset of V, then S is a subspace of V, if:

1) S is non-empty ($S \neq \emptyset$)
2) $\alpha x \in S$, for any $\alpha \in \mathbb{R}$ and $x \in S$ (closed under (·))
3) $x + y \in S$, for any $x, y \in S$ (closed under (+))

proof: We need to show that S is a vector space.

- C_1 & C_2 are satisfied (from (2) and (3)).
- A_1, A_2, ($A_5 - A_8$) are satisfied, because S is a subset of V.

- A_3: $\alpha x \in S$, take $\alpha = 0$, then $0 \cdot x \in S$
  $\Longrightarrow 0 \in S$

- A_4: $\alpha x \in S$, take $\alpha = -1$, then $-1 \cdot x \in S$
  $\Longrightarrow -x \in S$; $x + (-x) = 0$.

So, S is a vector space $\Longrightarrow$ S is a subspace of V.

A_7: $(\alpha\beta) \circ x = x^{\alpha\beta}$
$= (x^{\beta})^{\alpha}$
$= \alpha \circ (x^{\beta})$
$= \alpha \circ (\beta \circ x)$

A_8: $1 \circ x = x^1 = x$

So, W is a vector space.

X

ex1: Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , x_1 = -x_2 \right\}$ a subset of $\mathbb{R}^2$.

Is S a subspace of $V = \mathbb{R}^2$?

1) $S \neq \emptyset$, because $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in S$

2) Let α be a scalar , then :

$$\alpha x = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ -\alpha x_1 \end{pmatrix} \in S$$

3) Let $x = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} \in S$, $y = \begin{pmatrix} y_1 \\ -y_1 \end{pmatrix} \in S$

$$x + y = \begin{pmatrix} x_1 + y_1 \\ -x_1 - y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ -(x_1 + y_1) \end{pmatrix} \in S$$

So, S is a subspace of $V = \mathbb{R}^2$.

ex2: $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , x_2 + x_3 = 1 \right\}$, subset of $\mathbb{R}^3$.

Is W a subspace of $\mathbb{R}^3$?

1) $W \neq \emptyset$, since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in W$

2) take $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in W$, $\alpha = 2$, then :

$$\alpha x = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \notin W \text{ , since } x_2 + x_3 = 2 + 0 \neq 1$$

So, W isn't a subspace of $\mathbb{R}^3$.

note: if $0 \notin S$, then S can't be a subspace of V.

ex3: Let $S = \{0\}$ be a subset of some V.

Is S a subspace of V?

1) $S \neq \emptyset$, since $0 \in S$.

2) take $x = 0 \in S$, $\alpha \in \mathbb{R}$, then:

$\alpha x = \alpha(0) = 0 \in S$

3) take $x = 0$, $y = 0$, $x, y \in S$, then:

$x + y = 0 \in S \implies S$ is a subspace

note: the zero subspace of V is $S = \{0\}$.

ex4: $S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix}, \ a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2 \times 2}$

Is S a subspace of $\mathbb{R}^{2 \times 2}$?

1) $S \neq \emptyset$, because $x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$.

2) $x = \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in S$, $\alpha \in \mathbb{R}$, then

$\alpha x = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha a \end{bmatrix} \in S$

3) $\alpha = \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in S$, $y = \begin{bmatrix} d & e \\ f & d \end{bmatrix} \in S$, then:

$\alpha + y = \begin{bmatrix} \cdots & \\ c+f & a+d \end{bmatrix} \in S \Rightarrow S$ is a subspace

* Proper Subspace:

S is called a proper subspace of V, if $S \neq \{0\}$ and $S \neq V$.

note: V is a subspace of V.

* Null Space of A:

Def: Let A be an $(m \times n)$ matrix, then the null space of A is the subset that consists of all solutions to the homogeneous system $A\alpha = 0$.

Null space of A is denoted by $N(A)$.

$N(A) = \{\alpha \in \mathbb{R}^n ; A\alpha = 0\} \subseteq \mathbb{R}^n$

Is $N(A)$ a subspace of $\mathbb{R}^n$?

Answer: yes

Let $S = N(A)$, then the proof is as follows:

proof: 1) $S \neq \emptyset$, because $x = 0 \in S$ $(A.0 = 0)$.

2) Let $x \in S$, $\alpha \in \mathbb{R}$, then:

$x \in S \implies Ax = 0$

Now, $A(\alpha x) = \alpha(Ax) = \alpha(0) = 0 \in S$

So, $\alpha x \in S$.

3) Let $x \in S$, $y \in S$, then:

$x \in S \implies Ax = 0$
$y \in S \implies Ay = 0$

Now, $A(x+y) = Ax + Ay$
$= 0 + 0$
$= 0$

So, $x+y \in S$

So, $S = N(A)$ is a subspace of $\mathbb{R}^n$.

ex1: Find $N(A)$, if $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & -2 & 4 \end{bmatrix}$.

$Ax = 0$, we need to find x.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 3 & -2 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \end{array}\right] \quad (R_2 - 2R_1)$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \quad (R_2 * -1)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \quad (R_1 - 2R_1)$$

x_1 , x_2 : ~~free~~ leading variables

x_3 , x_4 : free variables

let $x_3 = t$, $x_4 = w$, $t, w \in \mathbb{R}$

$x_1 - x_3 - x_4 = 0 \Longrightarrow x_1 = t + w$

$x_2 + 2x_4 = 0 \Longrightarrow x_2 = -2w$

So, $x = \begin{pmatrix} t+w \\ -2w \\ t \\ w \end{pmatrix}$, $t, w \in \mathbb{R}$

$$N(A) = \left\{ \begin{pmatrix} t+w \\ -2w \\ t \\ w \end{pmatrix} ; \ t, w \in \mathbb{R} \right\}$$

note: we can write the solution as linear combination:

$$\begin{pmatrix} t+w \\ -2w \\ t \\ w \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Th: $A_{n \times n}$ is non-singular, iff $N(A) = \{0\}$.

proof: 1) Suppose that A is non-singular, then:

$Ax = 0$ has only the ~~non~~ zero solution

$$x = A^{-1} 0 = 0$$

$$\Rightarrow N(A) = \{0\}$$

2) Suppose that $N(A) = \{0\}$, then:

$x = 0$ is the only solution for $Ax = 0$

$\Rightarrow A$ is non-singular

So, A is non-singular, iff $N(A) = \{0\}$.

ex 2: $V = C[a,b]$

- $C^1[a,b]$ = all functions with continuous 1st derivative.

C^1 is a subspace of $C[a,b]$

- $C^2[a,b]$ = all functions with continuous 2nd derivative.

C^2 is a subspace of $C^1[a,b]$

and C^2 is a subspace of $C[a,b]$ also.

note: if V is a vector space and S, T are subspace of V and $S \subseteq T$, then S is a subspace of T

* In general, $C^n[a,b]$ is a subspace of $C[a,b]$

ex3: $V = C[-1,1]$ is a vector space, $S = C^1[-1,1]$

S is a proper subspace of V, because

$f(x) = |x| \in C[-1,1]$, because

$|x|$ is continuous on $[-1,1]$, but

$f(x) = |x| \notin C^1[-1,+1]$, because

$f(x) = |x|$ is not differentiable at $x=0$.

exH: $V = P_3 = \{ax^2+bx+c \ , \ a,b,c \in \mathbb{R}\}$

$S = \{p(x) \in P_3 \mid p(0) = 0\} \subseteq P_3$

Is S a subspace of P_3?

note that $S = \{p(x) \in P_3 \mid ax^2+bx \ , \ a,b \in \mathbb{R}\}$

1) $S \neq \emptyset$, since $p(0) = 0$

2) S is closed under $(+)$:

let $p(x), q(x) \in S$, then:

$p(0) = 0$, $q(0) = 0$

Now, $(p+q)(0) = p(0) + q(0)$
$= 0 + 0$
$= 0 \in S$

3) S is closed under $(\cdot)$:

let $p(x) \in S$, $\alpha \in \mathbb{R}$, then:

$p(0) = 0$

$(\alpha p)(0) = \alpha\left(p(0)\right) = \alpha(0) = 0 \in S$

So, S is a subspace of P_3.

* Spanning:

Def: Let V be a vector space and:

$v_1, v_2, \ldots, v_n \in V$

the linear combination of:

$v_1, v_2, \ldots, v_n$ is:

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, where $\alpha_i \in \mathbb{R}$, forall

Now, we define the span of $v_1, v_2, \ldots, v_2$ as follows:

span $(v_1, v_2, \ldots v_n)$ is all possible linear combinations of $v_1, v_2, \ldots, v_n$, that is:

$$\text{span}(v_1, v_2, \ldots v_n) = \{ z = \alpha_1 v_1 + \ldots + \alpha_n v_n \; ; \alpha_i \in \mathbb{R} \; ; \forall i$$

ex1: In $V = \mathbb{R}^3$, take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Find span (v_1).

$$\text{span}(v_1) = \{ \alpha v_1 \; ; \alpha \in \mathbb{R} \}$$

$$= \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \; ; \alpha \in \mathbb{R} \right\}$$

ex2: In $V = \mathbb{R}^4$, take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$; $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$.

Find span (v_1, v_2):

$$\text{span}(v_1, v_2) = \{\alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_1 \\ 3\alpha_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_2 \\ 2\alpha_2 \end{pmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 + \alpha_2 \\ 3\alpha_1 + 2\alpha_2 \end{pmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Th: span $(v_1, v_2, \dots, v_n)$ is a subspace of V.

proof: 1) $S \neq \emptyset$, since:

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$$

2) let $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in S$, α is scalar:

$$\alpha v = (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + \dots + (\alpha\alpha_n) v_n \in S$$

3) let $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in S$
$z = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \in S$

$$v + z = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \dots + (\alpha_n + \beta_n) v_n \in S$$

So, $S = \text{span}(v_1, v_2, \ldots, v_n)$ is a sub-space of V.

A spanning set of vector space (V).

ex1: In R^3, $v_1 = e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

$$\text{span}(e_1, e_2) = \{\alpha_1 e_1 + \alpha_2 e_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$= \left\{\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\\0\end{pmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$

Note that $\text{span}(e_1, e_2) \neq R^3$, so $\{e_1, e_2\}$ is not a spanning set for R^3.

ex2: In R^3, $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$.

$$\text{span}(e_1, e_2, e_3) = \{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

$$= \left\{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\}$$

$$= R^3.$$

So, $\{e_1, e_2, e_3\}$ is a spanning set for R^3.

ex3: In $\mathbb{R}^n$, $\{e_1, e_2, \dots, e_n\}$ is a spanning set for $\mathbb{R}^n$, prove that.

let $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$, then

~~span (V)~~ $V = a_1 e_1 + a_2 e_2 + \dots + a_n e_n = \mathbb{R}^n$

ex4: In $\mathbb{R}^3$, $\left\{ e_1, e_2, e_3, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}$

Is this a spanning set of $\mathbb{R}^3$?

let $v \in \mathbb{R}^3$, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a e_1 + b e_2 + c e_3 + 0 \cdot \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

So, it is a spanning set for $\mathbb{R}^3$.

note: if $\{v_1, v_2, \dots, v_n\}$ is a spanning set for V, then $\{v_1, v_2, \dots, v_n, v_{n+1}\}$ is also a spanning set.

ex5: In $\mathbb{R}^3$, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$

Is this a spanning set for R_3?

$$\text{span}(v_1, v_2) = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$\text{Span}(v_1, v_2) = \left\{ \begin{pmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \\ 2\alpha_1 + 4\alpha_2 \end{pmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Now, let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \\ 2\alpha_1 + 4\alpha_2 \end{pmatrix}$

$\Longrightarrow$

$$\alpha_1 + 3\alpha_2 = a \quad \text{--- (1)}$$
$$\alpha_1 + \alpha_2 = b \quad \text{--- (2)}$$
$$2\alpha_1 + 4\alpha_2 = c \quad \text{--- (3)}$$

$$\left[\begin{array}{cc|c} 1 & 3 & a \\ 1 & 1 & b \\ 2 & 4 & c \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & -2 & b-a \\ 0 & -2 & c-2a \end{array}\right] \quad \begin{array}{l} (R_2 - R_1) \\ (R_3 - 2R_1) \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & 1 & \frac{a-b}{2} \\ 0 & -2 & c-2a \end{array}\right] \quad (R_2 * \tfrac{-1}{2})$$

$$\left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & 1 & \frac{a-b}{2} \\ 0 & 0 & -a-b+c \end{array}\right] \quad (R_3 + 2R_2)$$

This system is inconsistent, iff $-a-b+c \neq 0$.

Take $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \Longrightarrow -a-b+c = -1-1+3 = 1 \neq 0$

So, this set isn't a spanning set of R^3.

ex: Show that the vectors x^2-1, $x+1$, $x+2$ span P_3.

proof: $P_3 = \{ax^2+bx+c \mid a,b,c \in R\}$

$$ax^2+bx+c = \alpha_1(x^2-1) + \alpha_2(x+1) + \alpha_3(x+2)$$
$$= \alpha_1 x^2 - \alpha_1 + \alpha_2 x + \alpha_2 + \alpha_3 x + 2\alpha_3$$
$$ax^2+bx+c = \alpha_1 x^2 + (\alpha_2+\alpha_3)x + (-\alpha_1+\alpha_2+2\alpha_3)$$

So,
$a = \alpha_1$ --- (1)
$b = \alpha_2 + \alpha_3$ --- (2)
$c = -\alpha_1 + \alpha_2 + 2\alpha_3$ --- (3)

$a+c = \alpha_2 + 2\alpha_3$
$a+c = (b-\alpha_3) + 2\alpha_3$
$a+c = b + \alpha_3$
$\Rightarrow \boxed{\alpha_3 = a+c-b}$

$b = \alpha_2 + \alpha_3$
$b = \alpha_2 + (a+c-b)$
$\Rightarrow \boxed{\alpha_2 = 2b-a-c}$

$\boxed{\alpha_1 = a}$

So, the vectors are span for P_3.

ex) x^2+3

$a=1$, $b=0$, $c=3$
$\Rightarrow \alpha_1 = 1$, $\alpha_2 = -4$, $\alpha_3 = 4$

ex7: $1, x$ in P_3. Is $\{1, x\}$ a spanning set for P_3?

$ax^2 + bx + c = \alpha_1(1) + \alpha_2(x) + 0(x^2)$

So, $a = 0$
$b = \alpha_2$
$c = \alpha_1$

$x^2 \neq \alpha_1(1) + \alpha_2(x)$

x^2 cannot be written as a linear combination of 1 and x. So, $\{1, x\}$ isn't a spanning set for P_3.

ex8: Is $\{3, x^2\}$ a spanning set for P_3?

$ax^2 + bx + c = \alpha_1(x^2) + \alpha_2(0) + \alpha_3(3)$

So, $a = \alpha_1$
$b = 0$
$c = 3\alpha_3$

x cannot be written as a linear combination of 3 and x^2. So, $\{3, x^2\}$ isn't a spanning set for P_3

Remark: If span $(v_1, v_2, \ldots, v_n) = V$, then any vector in V can be written as a linear combination of $v_1, v_2, \ldots, v_n$.

(3.3) Linear Independence

Def: let $v_1, v_2, \dots, v_n$ be vectors in a vector space V, then we say that $v_1, v_2, \dots, v_n$ are linearly independent, if the homogeneous equation:

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad , \; c_i \in \mathbb{R} \; , \forall i$$

has only the trivial solution. We say that $v_1, v_2, \dots, v_n$ are linearly dependent, if the homogeneous equation:

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

has a non-trivial solution ($c_i \neq 0$, for ~~all i~~ any i)

ex1: In $\mathbb{R}^3$, Is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

linearly independent or linearly dependent?

$$c_1 + c_2 = 0$$
$$2c_1 + 2c_2 + c_3 = 0$$
$$3c_1 + c_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} (R_2 - 2R_1) \\ (R_3 - 3R_1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad (R_2 \longleftrightarrow R_3) \text{ then } (R_2 * \tfrac{-1}{3})$$

$\Longrightarrow$ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ (no free variables)
$\Longrightarrow$ they are linearly independent.

ex2: Is $\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{V_1}, \underbrace{\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}}_{V_2} \right\}$ linearly independent?

$V_2 = 2V_1 \Longrightarrow 2V_1 - V_2 = 0$
$\Longrightarrow c_1 = 2$, $c_2 = -1$ (~~not~~ non-zero solution
$\Longrightarrow$ linearly dependent

or we can solve the following system to see if there are any free variables:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, c_2 is a free variable $\Longrightarrow$ V_1 & V_2 are L.D.

note: if v_1, v_2 are vectors in V, then v_1, v_2 are L.D, iff one of them is a multiple of the other one.

ex3: In R^2, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

test weather v_1, v_2 and v_3 are linearly dependent or linearly independent.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2c_3 \\ c_3 \end{pmatrix} = 0$$

$$\begin{bmatrix} c_1 & c_2 & 2c_3 \\ c_1 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $c_1 + c_2 + 2c_3 = 0$
$c_1 + c_3 = 0$

$\Rightarrow$ underdetermined homogeneous system.
$\Rightarrow$ has infinitely many solutions
$\Rightarrow$ there is a non-trivial solution.
$\Rightarrow$ v_1, v_2, v_3 are L.D.

note: In R^n, if $v_1, v_2, \ldots, v_K$ are vectors in R^n with $K > n$, then $v_1, v_2, \ldots, v_K$ are L.D.

Th: If $V_1, V_2, \ldots, V_n$ are vectors in R^n, then $V_1, V_2, \ldots, V_n$ are L.I, iff the matrix $U = (V_1, V_2, \ldots, V_n)$ is non-singular

proof: $c_1 V_1 + c_2 V_2 + \ldots + c_n V_n = 0$

$$\Longrightarrow \quad U_{n\times n} \, C_{n\times 1} = 0_{n\times 1}$$

So, $UC = 0$ has only the trivial solution, iff U is non-singular.

Thus, $V_1, V_2, \ldots, V_n$ are L.I, iff U is non-singular.

ex4: Test weather the following vectors are L.D or L.I.

$$V_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 7 \end{bmatrix}$$

$$|U| = 8 \neq 0$$

So, U is non-singular $\Longrightarrow V_1, V_2, V_3$ are L.I.

ex 5: Test weather the vectors:

x^2-2, $x+4$, 5

are L.I or L.D.

$c_1(x^2-2) + c_2(x+4) + c_3(5) = 0$

$c_1x^2 - 2c_1 + c_2x + 4c_2 + 8c_3 = 0\cdot x^2 + 0\cdot x + 0$

$c_1(x^2) + c_2(x) + (-2c_1 + 4c_2 + 5c_3) = 0\cdot x^2 + 0\cdot x + 0$

$\Rightarrow c_1 = 0, \; c_2 = 0, \; 5c_3 + 4c_2 - 2c_1 = 0$

So, $c_1 = c_2 = c_3 = 0$

$\Rightarrow$ L.I.

ex 6: x, x^3 in P_4, are they L.I or L.D?

x & x^3 are not multiple of each other, so they are L.I.

or we can use the Def:

$c_1(x) + c_2(x^3) = 0\cdot(x^3) + 0\cdot(x^2) + 0\cdot(x) + 0$

So, $c_1 = 0$, $c_2 = 0$ unique solution (no free variables)

So, they are L.I.

ex7: $P_1 = x^2 - 2x + 3$, $P_2 = 2x^2 + x + 8$, $P_3 = x^2 + 8x + 7$
are those vectors L.I or L.D?

$c_1 P_1 + c_2 P_2 + c_3 P_3 = 0$

$c_1 (x^2 - 2x + 3) + c_2 (2x^2 + x + 8) + c_3 (x^2 + 8x + 7)$
$= 0 \cdot x^2 + 0 \cdot x + 0$

So, $c_1 + 2c_2 + c_3 = 0$
$-2c_1 + c_2 + 8c_3 = 0$
$3c_1 + 8c_2 + 7c_3 = 0$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 1 & 8 & 0 \\ 3 & 8 & 7 & 0 \end{array}\right]$$

$$\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = 0 \Rightarrow$$ there is a non-trivial solution

So, those vectors are L.D.

ex8: $x, x^2, 2x, 1$ are 4 vectors in P_3 , Test if L.I.

$V_3 = 2x$ is a multiple of $V_1 = x$

So, they are L.D.

note: If $P_1, P_2, \dots, P_K$ are vectors in P_n & $K>n$, then $P_1, P_2, \dots, P_K$ are L.D.

ex 9: Take $V_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $V_2 = \begin{pmatrix}2\\2\end{pmatrix}$ in R^2.

Clearly, they are L.D, since $V_2 = 2V_1$.

$$\begin{pmatrix}3\\3\end{pmatrix} \in \text{span}\Big(\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{V_1}, \underbrace{\begin{pmatrix}2\\2\end{pmatrix}}_{V_2}\Big)$$

$$\begin{pmatrix}3\\3\end{pmatrix} = (1)\,V_1 + (1)\,V_2$$

$$= (3)\,V_1 + (0)\,V_2$$

$$= (-1)\,V_1 + (2)\,V_2$$

Notice that $\begin{pmatrix}3\\3\end{pmatrix} \underset{\text{span}}{\in} \Big(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\Big)$ can be written in many ways, that is the representation of $\begin{pmatrix}3\\3\end{pmatrix}$ is not unique.

Th: Let $v_1, v_2, v_3, \dots, v_n$ be vectors in a vector space V and take $u \in \text{Span}(v_1, v_2, \dots, v_n)$, then $v_1, \dots, v_n$ are L.I, iff u can be written uniquely as a linear combination of $v_1, v_2, \dots, v_n$.

proof: 1) Assume $v_1, v_2, \dots, v_n$ are L.I and

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad --- (1)$$

Now, suppose that u has another representation, so we say:

$$u = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \quad --- (2)$$

we need to show that $\alpha_i = \beta_i$, for all i.

From (1) and (2);

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

$$(\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots + (\alpha_n - \beta_n) v_n = 0$$

But $v_1, v_2, \dots, v_n$ are L.I.

So, $\alpha_i - \beta_i = 0 \quad , \forall i$

$$\Longrightarrow \alpha_i = \beta_i \quad , \forall i$$

2) Assume any vector u in span $(v_1, v_2, \dots, v_n)$ has a unique ~~solution~~ representation.

we need to show that $v_1, v_2, \dots, v_n$ are L.I

So, using contradiction:

Assume $v_1, v_2, \dots, v_n$ are L.D, then there are $c_1, c_2, \dots, c_n$ not all zeros, such that:

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

take $u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + (0)$$

$$\Rightarrow u = \alpha_1 v_1 + \dots + \alpha_n v_n + (c_1 v_1 + \dots + c_n v_n)$$

$$u = (\alpha_1 + c_1) v_1 + (\alpha_2 + c_2) v_2 + \dots + (\alpha_n + c_n) v_n$$

But $c_1, c_2, \dots, c_n$ not zeros, that is:

$$\alpha_i + c_i \neq \alpha_i, \text{ for all } i$$

$\Rightarrow$ u in span $(v_1, v_2, \dots, v_n)$ doesn't has a unique representation.

$\Rightarrow$ Contradiction.

So, our assumption is wrong, that is $v_1, v_2, \dots, v_n$ are L.I.

~~(3.3)~~ * Wronskian:

Def: let $f_1, f_2, \dots, f_n$ be elements in $C^{(n-1)}[a,b]$, then the wronskian of $f_1, f_2, \dots, f_n$ is a function denoted by $W(f_1, f_2, \dots, f_n)$, such that:

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

Result: 1) If $f_1, f_2, \dots, f_n$ are L.D in $C^{(n-1)}[a,b]$, then $W(f_1, f_2, \dots, f_n)(x) \equiv 0$ for all x in $[a,b]$.

note: " $\equiv 0$ " means identically zero.

2) If $W(f_1, f_2, \dots, f_n) \neq 0$ for at least one x_0 in $[a,b]$, then $f_1, f_2, \dots, f_n$ are L.I.

note: If $W(f_1, f_2, \dots, f_n) = 0$, we cannot tell any thing.

ex1: $f(x) = e^x$, $g(x) = e^{-x}$ in $C(-\infty, \infty)$, test weather f & g are L.D or L.I.

$$W(f, g) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^0 - e^0$$
$$= -2 \neq 0$$

So, f & g are L.I

ex2: $f(x) = 1$, $g(x) = x$, $h(x) = x^2$ in $C[-\infty, \infty]$.
Test if they are L.I or LD.

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

So, they are L.I.

ex3: $f(x) = e^x$, $g(x) = e^{x+2}$ in $C[-\infty, \infty]$, check if L.I.

$$W(f, g) = \begin{vmatrix} e^x & e^{x+2} \\ e^x & e^{x+2} \end{vmatrix} = e^{2x+2} - e^{2x+2} = 0$$

So, we cannot tell any-thing.

But clearly, they are L.D, since:
$g(x) = e^{x+2} = e^2 . e^x = e^2 . f(x)$
(they are multiple of each others)

Also, take $c_1 e^x + c_2 e^{x+2} = 0$, then we find that $c_1 = e^2$, $c_2 = -1$ (non-trivial solution).

ex4: $f(x) = x^2$, $g(x) = x|x|$, show that f & g are L.I in $C'[-1,1]$.

$$W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0$$

So, we cannot tell anything.

Back to definition:

$$c_1 x^2 + c_2 x|x| = 0 \quad , \quad x \in [-1,1]$$

~~take $x=1 \Rightarrow c_1 + c_2 = 0$~~

take $x = -1 \Rightarrow c_1 - c_2 = 0$

$2c_1 = 0 \Rightarrow \boxed{c_1 = 0}$

$c_1 = 0 \Rightarrow \boxed{c_2 = 0}$

$\Rightarrow$ There is a trivial solution for two values

So, they are L.I.

ex5: $f(x) = \sin x$, $g(x) = \cos x$ in $C(-\infty, \infty)$, check for dependence:

$$W(f,g) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x) = -1 \neq 0$$

So, f & g are L.I.

ex6: $f(x) = x^2$, $g(x) = x|x|$, in $C[0,1]$.

f & g are L.D, because for $x \in [0,1]$, $|x| = x$, that is:

$$g(x) = x \cdot x = x^2 = f(x).$$

ex7: $f(x) = 2x$, $g(x) = |x|$, show that f & g are L.I in $C[-1,1]$.

$$W(2x, |x|) = \begin{vmatrix} 2x & |x| \\ 2 & (|x|)' \end{vmatrix}$$

But $|x| \notin C^1[-1,1]$, since $|x|$ isn't differentiable at $x=0$, so we can't use the wronskian.

Using definition:

$$c_1(2x) + c_2(|x|) = 0$$

take $x = 1 \Rightarrow 2c_1 + c_2 = 0$
take $x = -1 \Rightarrow -2c_1 + c_2 = 0$

$$2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$$

$$c_2 = 0 \Rightarrow \boxed{c_1 = 0}$$

There is a trivial solution for two values.

So, they are L.I.

note: f & g are L.D in $C[0,1]$, because $g(x) = |x| = x = \frac{1}{2} f(x)$.

(3.4) Basis and Dimensions

Def: If $v_1, v_2, \dots, v_n$ are vectors in a vector space V, then the set $\{v_1, v_2, \dots, v_n\}$ is called a basis

1) $v_1, v_2, \dots, v_n$ span V ($\text{span}(v_1, v_2, \dots, v_n) = V$) (i.e. $\{v_1, v_2, \dots, v_n\}$ is a spanning set for V.)

2) $v_1, v_2, \dots, v_n$ are L.I.

Result: 1) $\{v_1, v_2, \dots, v_n\}$ is a basis ~~iff~~ for V, iff every vector in V can be written uniquely as a linear combination of $v_1, \dots, v_n$.

Th: If $v_1, v_2, \dots, v_n$ span V and one of these vectors can be written as a linear combina of the others, then the other vectors form a spanning set for V.

proof: suppose that $v_1, v_2, \dots, v_n$ is a spanning set for V.

assume v_n can be written as a linear combination of the other vectors $(v_1, v_2, \dots, v_{n-1})$

So, $v_n = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{n-1} v_{n-1}$

We need to show that any vector ~~v~~ in V can be written as a linear combina of $(v_1, v_2, \dots, v_{n-1})$.

Since $v_1, v_2, \dots, v_n$ is a spanning set for V,
then:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_{n-1} v_{n-1} + c_n v_n$$

$$= c_1 v_1 + c_2 v_2 + \dots + c_{n-1} v_{n-1} + c_n (\alpha_1 v_1 + \dots + \alpha_{n-1} v_{n-1})$$

$$= (c_1 + \alpha_1 c_n) v_1 + (c_2 + \alpha_2 c_n) v_2 + \dots + (c_{n-1} + c_n \alpha_{n-1}) v_{n-1}$$

$$= B_1 v_1 + B_2 v_2 + \dots + B_{n-1} v_{n-1}$$

$\therefore$ So, $v_1, v_2, \dots, v_{n-1}$ is a spanning set for V.

2) The set $\{v_1, v_2, \dots, v_n\}$ is a basis for V, iff $\{v_1, v_2, \dots, v_n\}$ is a minimal (smallest) spanning set.

ex1: $V = \mathbb{R}^3$, $B = \{e_1, e_2, e_3\}$.

Is B a basis for $V = \mathbb{R}^3$?

Yes, B is a spanning set, because:

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a e_1 + b e_2 + c e_3$$

and B is L.I, because:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

ex2: $V = \mathbb{R}^2$, $B = \{e_1, e_2\}$.

Is B a basis for $V = \mathbb{R}^2$?

Yes, B is a spanning set, because:

$$v = \begin{pmatrix} a \\ b \end{pmatrix} = a e_1 + b e_2$$

and B is L.I., since $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$.

~~ex3:~~

note: In general, in $\mathbb{R}^n$, $B = \{e_1, e_2, \dots, e_n\}$ is a basis for $\mathbb{R}^n$ & called "Standard Basis".

ex 3: In $\mathbb{R}^3$, $S = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{V_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{V_2}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{V_3} \right\}$.

Is S a basis for $\mathbb{R}^3$?

Yes, because:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{L.I}$$

and S is a spanning set for $\mathbb{R}^3$, since:

$$\operatorname{span}(v_1, v_2, v_3) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 \end{pmatrix}$$

take $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 \end{pmatrix}$, we will get

$\alpha_1 = c$, $\alpha_2 = b - c$, $\alpha_3 = a - b$

$\Rightarrow \{v_1, v_2, v_3\}$ are a basis for $\mathbb{R}^3$.

note: in general, any basis for $\mathbb{R}^n$ must consist exactly from n vectors.

ex 4: In P_3, take $B = \{1, x, x^2\}$.

Is B a basis for P_3?

Yes, since $1, x, x^2$ are L.I, because:

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

and since $B = \{1, x, x^2\}$ is a spanning set for P_3 because:

$$v = ax^2 + bx + c = c(1) + b(x) + a(x^2)$$

note: in general, for P_n, the set $\{1, x, x^2, \dots, x^{n-1}\}$ is called the standard basis for P_n.

ex5: In P_3, the set $S = \{x+1, x+2, x^2+1\}$ is also a basis for P_3, because S is a spanning set and it is independent.

proof:
$$\begin{vmatrix} x+1 & x+2 & x^2+1 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 0 - 0 + 2\begin{vmatrix} x+1 & x+2 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$$

$\Rightarrow$ L.I

take $ax^2 + bx + c = \alpha_1(x+1) + \alpha_2(x+2) + \alpha_3(x^2+1)$

$ax^2 + bx + c = \alpha_3 x^2 + (\alpha_1 + \alpha_2)x + (\alpha_1 + 2\alpha_2 + \alpha_3)$

$\Rightarrow a = \alpha_3, \quad b = \alpha_1 + \alpha_2, \quad c = \alpha_1 + 2\alpha_2 + \alpha_3$

$\Rightarrow \alpha_1 = 2b + a - c, \quad \alpha_2 = c - a - b, \quad \alpha_3 \; a$

$\Rightarrow$ they are a spanning set.

So, they are a basis for P_3.

note: For P_n, any basis ~~may~~ must consist of n vectors

ex6: $V = \mathbb{R}^{2*3}$, $B = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$

$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\quad E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Show that B is a basis for $\mathbb{R}^{2*3}$.

proof: $E_{11}, \ldots, E_{23}$ are L.I, because:

$c_1 E_{11} + c_2 E_{12} + c_3 E_{13} + c_4 E_{21} + c_5 E_{22} + c_6 E_{23} = 0$

So, $\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

~~since~~ $\Rightarrow c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$

Now, $E_{11}, \ldots, E_{23}$ span $\mathbb{R}^{2\times 3}$, to prove that:

take $v = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$= a E_{11} + b E_{12} + c E_{13}$
$+ d E_{21} + e E_{22} + f E_{23}$

So, B is a basis for $\mathbb{R}^{2\times 3}$.

note: any basis for $\mathbb{R}^{m\times n}$ must consist of $m \times n$ vectors.

ex7: Let $T = \left\{ \begin{pmatrix} a+b \\ a \\ b \end{pmatrix}, a, b \in \mathbb{R} \right\}$

is a subspace of $\mathbb{R}^3$. Find a basis for T.

$$\begin{pmatrix} a+b \\ a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{V_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{V_2} \right\}$$

So, $V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are span(T).

Also, V_1 & V_2 are L.I because they aren't multiple of each other.

So, $B = \{V_1, V_2\}$ is a basis for T.

ex8: $W = \{ P(x) \in P_3 , P(0) = 0 \}$ is a subspace of

Find a basis for W.

$W = \{ ax^2 + bx + c , P(0) = 0 \}$

$P(0) = 0 \Rightarrow a(0)^2 + b(0) + c = 0$

$\Rightarrow \boxed{c = 0}$

$\Rightarrow W = \{ ax^2 + bx ; P(0) = 0 , a, b \in \mathbb{R} \}$

Now $ax^2 + bx = a\underbrace{(x^2)}_{v_1} + b\underbrace{(x)}_{v_2}$

So, $ax^2 + bx = av_1 + bv_2 = \text{span}(x, x^2)$

So, x, x^2 is span (W).

Also, x & x^2 are L.I, because:

Both x, x^2 are not multiple of each other.

So, $B = \{x, x^2\}$ is a basis for W.

Th (*): If $v_1, v_2, \dots, v_n$ span a vector space V, then any collection of m vectors $u_1, u_2, \dots, u_m$, where $m > n$, must be be L.D.

ex: $V = \mathbb{R}^3$, $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$.

v_1, v_2, v_3 are span $\mathbb{R}^3$.

take $u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $u_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $u_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$, $u_4 = \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix}$

then by Th (*), u_1, u_2, u_3, u_4 are L.D.

Result: If $v_1, v_2, \ldots, v_n$ are span V and $u_1, u_2, \ldots, u_m$ are L.I set in V, then what can we say about m?

We can say that $m \leq n$, since if $m > n$, then $u_1, u_2, \ldots, u_m$ will be L.D.

Th: If $B = \{v_1, v_2, \ldots, v_n\}$ and $S = \{u_1, u_2, \ldots, u_m\}$ are both bases for a vector space V, then $m = n$

proof: Since $\{v_1, v_2, \ldots, v_n\}$ is a basis, then $v_1, v_2, \ldots, v_n$ span V. Also $u_1, u_2, \ldots, u_m$ are L.I, because S is a basis. So, by the previous result $\boxed{m \leq n}$ --- (1)

Similarly, $u_1, u_2, \ldots, u_m$ span V and $v_1, v_2, \ldots, v_n$ are L.I $\Longrightarrow \boxed{n \leq m}$ --- (2)

From (1) and (2), $m = n$.

Def: Let V be a vector space and assume $v_1, v_2, v_3, \ldots, v_n$ form a basis for V, then we can define what so called the dimension of V, denoted by $\dim(V)$, as the number of the elements in the basis, and we say $\dim(V) = n$.

Def: If V has a dimension n, we say that V is finite dimensional, otherwise V is called infinite dimensional.

Def: The zero vector space $V = \{0\}$ has a dimension equals zero.

examples of finite dimensional vector spaces:

1) $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$.

2) P_n, $\dim(P_n) = n$.

3) $\mathbb{R}^{m*n}$, $\dim(\mathbb{R}^{m*n}) = m*n$

examples of infinite dimensional vector spaces:

1) $C[a,b]$, 2) P: all polynomials.

proof of (2) [by contradiction]:

Assume $\dim(P) = n$, so there n vectors that are L.I and spanning P.

Consider the set $\{1, x, x^2, \dots, x^n\}$.

This set consists of $(n+1)$ vectors, so by Th (*): $1, x, x^2, \dots, x^n$ are L.D, but that isn't right, because $1, x, x^2, \dots, x^n$ are L.I, since $W(1, x, x^2, \dots, x^n) \neq 0$.

Contradiction $\Longrightarrow$ $\dim(P) \neq n$, that is P is infinite dimensional.

Th: If V is a vector space with $\dim(V) = \underline{\underline{n}} > 0$, then:

1) Any $\underline{n}$ L.I vectors must span.
2) Any spanning set of $\underline{n}$ vectors must be L.I.

ex: $V = \mathbb{R}^3$, take $B = \left\{ \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}_{v_3} \right\}$

Is B a basis for V?

$\dim(V) = 3$ & the set consists of 3 vectors so it is suffices to show that B is L.I set

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = -3 \neq 0 \Rightarrow \text{L.I}$$

$\Rightarrow$ spanning set $\Rightarrow$ B is basis for $V = \mathbb{R}^3$

* Result: If $\dim(V) = n > 0$, then no set of fewer than n vectors can span V.

* Remark: take $V = \mathbb{R}^n$, $B = \{v_1, v_2, v_3, \dots, v_K\}$

1) if $K < n$, then B cannot span V.
2) if $K > n$, then B is L.D.
3) if $K = n$, then we can't tell anything. (we should check at least one condition from the two previous conditions, since we know that $\dim(V) = n$)

Th: If $\dim(V) = n > 0$, then:

1) Any set of K L.I vectors with $K < n$ can be extended to form a basis.

2) Any spanning set of K vectors with $K > n$, can be shrinked to form a basis.

ex: $x_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $x_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$, $x_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Are spanning set of $\mathbb{R}^3$, form a basis from these vectors.

First, we should choose two L.I vectors (not multiple of each other), so let's take:

$x_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$ which are L.I

Now take a third vector and check for independence:

$$|x_1 \; x_2 \; x_3| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 4 & 2 \end{vmatrix} = 0 \Rightarrow \text{L.D}$$

$$|x_1 \; x_2 \; x_4| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 4 & 4 \end{vmatrix} = 0 \Rightarrow \text{L.D}$$

$$\begin{vmatrix} x_1 & x_2 & x_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 5 & 1 \\ 4 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= -4 - -4 + -2$$

$$= -2 \neq 0$$

So, $\{x_1, x_2, x_5\}$ are L.I.

$\Longrightarrow$ they can form a basis for $\mathbb{R}^3$.

(3.5) Change of Basis.

* Recall: If $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V, then any vector $v \in V$ can be written uniquely as a linear combination of $v_1, v_2, \dots, v_n$:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$\Rightarrow$ The scalars $(\alpha_1, \alpha_2, \dots, \alpha_n)$ are called the coordinates of v with respect to the basis B.

Moreover, we can define the coordinate vector of v, as:

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

read: ~~with~~ the coordinate vector of v with respect to basis B.

ex1: $V = \mathbb{R}^2$, standard basis $B = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

take $v = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, find $[v]_B$.

$$v = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \underset{\alpha_1}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underset{\alpha_2}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

ex2) $V = \mathbb{R}^2$, $B' = \{e_2, e_1\}$ (orderd basis)

take $v = |5|$, find $[v]_{B'}$.

$$v = \begin{pmatrix}5\\4\end{pmatrix} = \underset{\alpha_1}{4}\ \underset{v_1}{e_2} + \underset{\alpha_2}{5}\ \underset{v_2}{e_1}$$

So, $[v]_{B'} = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$.

ex3) $V = \mathbb{R}^2$, $B = \left\{ v_1 = \begin{pmatrix}2\\3\end{pmatrix} , v_2 = \begin{pmatrix}2\\4\end{pmatrix} \right\}$

1) $v = \begin{pmatrix}0\\1\end{pmatrix}$, find $[v]_B$.

2) if $[v]_B = \begin{pmatrix}3\\4\end{pmatrix}$, find v.

1) $$v = \begin{pmatrix}0\\1\end{pmatrix} = \underset{\alpha_1}{(-1)} \underbrace{\begin{pmatrix}2\\3\end{pmatrix}}_{v_1} + \underset{\alpha_2}{(1)} \underbrace{\begin{pmatrix}2\\4\end{pmatrix}}_{v_2}$$

So, $[v]_B = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

2) $[v]_B = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$

$\Longrightarrow$ $v = \alpha_1 v_1 + \alpha_2 v_2$

$v = 3\begin{pmatrix}2\\3\end{pmatrix} + 4\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix}$

Q1 V is a vector space with $\dim(V) = n > 0$ with 2 bases, $B = \{v_1, v_2, \dots, v_n\}$ and $B' = \{u_1, u_2, \dots, u_n\}$.

1) if $[v]_B$ is given, then what is $[v]_{B'}$?
2) if $[v]_{B'}$ is given, then what is $[v]_B$?

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$$\Rightarrow [v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$v = \gamma_1 u_1 + \gamma_2 u_2 + \dots + \gamma_n u_n$

$$\Rightarrow [v]_{B'} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

We need to use a special $n \times n$ non-singular matrix called the transition matrix $S_{n \times n}$:

For (1): $S = \left([v_1]_{B'} \; [v_2]_{B'} \; \dots, \; [v_n]_{B'} \right)_{n \times n}$

$\Rightarrow [v]_{B'} = S\,[v]_B$

For (2): the transition matrix is S^{-1}

$\Rightarrow [v]_B = S^{-1}\,[v]_{B'}$

* Remark: $V = \mathbb{R}^n$, $E = \{e_1, e_2, \cdots, e_n\}$ (standard basis)

taking $v \in V$, $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, then $[v]_E = v$,

because $v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$.

ex: $V = \mathbb{R}^3$, $E = \{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{e_3} \}$

take $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 e_1 + 2 e_2 + 3 e_3$

$\Longrightarrow [v]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = v$

ex 1: $V = \mathbb{R}^2$, $E = \{e_1, e_2\}$, $B = \{ v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$

$[v]_B$ is given , find $[v]_E$.

assume $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, then:

$[v]_E = v = \alpha_1 v_1 + \alpha_2 v_2$

$[v]_E = (v_1 \ v_2) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$[v]_E = S \ [v]_B$

So, the trasition matrix from B to E is $S = (v_1\ v_2)$,
that is, $S = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$.

The transition matrix from E to B is $S^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}^{-1}$.

$$S^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{bmatrix}$$

$$\Rightarrow [v]_E = S\,[v]_B \quad , \quad [v]_B = S^{-1}\,[v]_E$$

ex2: $V = \mathbb{R}^2$, $E = \{e_1, e_2\}$, $B = \left\{ v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

use S & S^{-1} from the previous example to find:

1) v, if $[v]_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. 2) $[v]_B$, if $v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

1) $v = [v]_E = S\,[v]_B$

$$v = [v]_E = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$$

to check that:

$$2v_1 + 5v_2 = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix} \Rightarrow \text{true}$$

2) $$[v]_B = S^{-1}[v]_E \qquad ([v]_E = v)$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ & \end{bmatrix} \begin{bmatrix} 1 \\ \; \end{bmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}$$

to check that:

$$v = \frac{4}{5}\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \text{true}$$

※ $V = \mathbb{R}^n$, $B = \{u_1, u_2, \dots, u_n\}$, $B' = \{w_1, w_2, \dots, w_n\}$ are 2 bases for V, find transition matrix from B to B' and vice versa.

take $v \in V$, then:

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = \gamma_1 w_1 + \gamma_2 w_2 + \dots + \gamma_n w_n$$

$$(u_1 \; u_2 \; \dots \; u_n)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = (w_1 \; w_2 \; \dots \; w_n)\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$U\,[v]_B = W\,[v]_{B'}$$

$$[v]_B = \underbrace{U^{-1} W}_{S}\,[v]_{B'}$$

So, S is the transition matrix from B' to B, and

S^{-1} is the transition matrix from B to B'.

ex: $V = \mathbb{R}^2$, $B = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$B' = \left\{ w_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, w_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \right\}$

are 2 bases for V.

1) Find transition matrix from B to B'.
2) If $v = 2u_1 + 3u_2$, find $[v]_{B'}$.

1) $v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = \gamma_1 w_1 + \gamma_2 w_2 + \dots + \gamma_n w_n$

$$(u_1 \ u_2 \ \dots \ u_n) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = (w_1 \ w_2 \ \dots \ w_n) \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$U [v]_B = W [v]_{B'}$$

$$\Longrightarrow [v]_{B'} = \underbrace{W^{-1} U}_{S} [v]_B$$

So, $S = W^{-1} U = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \frac{1}{-1} \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 3 & 4 \end{bmatrix}$$

2) $V = 2u_1 + 3u_2 \Longrightarrow [v]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$[v]_{B'} = U\,[v]_B$$

$$= \begin{bmatrix} 8 & 11 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 49 \\ 18 \end{bmatrix}$$

to check that,

$$V = 49\,w_1 + 18\,w_2 = 49 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 18 \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

But $v = 2u_1 + 3u_2 = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

So, it is true.

ex: $V = P_3$, $E = \{1, x, x^2\}$, $B = \{1, 1+x, 1+x+x^2\}$

1) Find trasition matrix from E to B.
2) Find transition matrix from B to E.

1) $S_{E \to B} = \left([1]_B \; [x]_B \; [x^2]_B \right)$

Now, $[1]_B = ??$

$1 = 1(1) + 0(1+x) + 0(1+x+x^2)$

$\Rightarrow [1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$[x]_B = ??$

$x = -1(1) + 1(1+x) + 0(1+x+x^2)$

$\Rightarrow [x]_B = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$[x^2]_B = ??$

$x^2 = 0(1) + -1(1+x) + 1(1+x+x^2)$

$\Rightarrow [x^2]_B = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

So, $S_{E \to B} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

2) $S^{-1}_{B \to E} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

(3.6) Row Space & Column Space

Let A_{m*n} be a $(m*n)$ matrix, then we have three important subspaces for A:

1) Null space of A, denoted by $N(A)$.
2) Row space of A, denoted by $RS(A)$.
3) Column space of A, denoted by $CS(A)$.

1) Null space $(N(A))$ is subspace of $\mathbb{R}^n$

$$N(A) = \{ x \in \mathbb{R}^n ; \; A_{m*n} x_{n*1} = 0 \}$$

2) $RS(A)$ is all linear combinations of the row vectors of A.

$RS(A)$ = span of rows

$RS(A)$ is a subspace of $\mathbb{R}^{1*n}$.

3) $CS(A)$ is all linear combination of column vector of A.

$CS(A)$ = span of columns.

$CS(A)$ is a subspace of $\mathbb{R}^m$.

ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Find $RS(A)$, $CS(A)$.

1) $RS(A) = \{ \alpha_1 (1,0,0) + \alpha_2 (0,1,0) \; ; \; \alpha_1, \alpha_2 \in \mathbb{R} \}$

$RS(A) = \{ (\alpha_1, \alpha_2, 0) \; ; \; \alpha_1, \alpha_2 \in \mathbb{R} \}$

$RS(A)$ is subspace of $\mathbb{R}^{1 \times 3}$

2) $CS(A) = \{ \gamma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \gamma_3 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; ; \; \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \}$

$CS(A) = \{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \; ; \; \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \}$

$CS(A)$ is subspace of $\mathbb{R}^2$ & equal $\mathbb{R}^2$.

★ How to find a basis for $RS(A)$, $CS(A)$?

Def: Dimension of $N(A)$ called Nullity.

Def: Dimension of $RS(A)$ = Dimension of $CS(A)$
= Rank (A)

Th: Any two row equivalent matrices have the sam row space.

proof: If A, B are row equivalent, then A can be obtained from B using elementary row operations, So, $RS(A) \subseteq RS(B)$.

Similarly, since B is row equivalent to A, then B can be obtained from A using elementary row operations, So, $RS(B) \subseteq RS(A)$.

since $RS(A) \subseteq RS(B)$ and $RS(B) \subseteq RS(A)$, then $RS(A) = RS(B)$.

* To find RS(A), transform A to REF(A) (or RREF) and the non-zero rows of REF(A) is a basis for RS(REF(A)) and consequently is a basis for RS(A)

ex1: $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

Find a basis for RS(A), then find Rank(A).

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis of RS(A) is $\{(1,0,-1), (0,1,1)\}$

Rank (A) = 2.

Result: Rank (A) = the number of nonzero rows in REF(A).
= the number of the leading ones in REF (A).

ex2: Find a basis for CS (A), if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$REF(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = u$$

$$\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}$$

Basis for CS (REF(A)) is $\left\{ \overset{u_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \right\}$.

So, $\{a_1, a_2\}$ is a basis for CS (A)

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$ is basis for CS (A).

note: $u_3 = -u_1 + u_2$ & $a_3 = -a_1 + a_2$

So, columns of A & columns of REF(A) have the same dependency relation.

Th : $A_{m \times n}$, then, $\text{Rank}(A) \leq \min(m,n)$.

$\text{Rank}(A) =$ number of non-zero rows in $\text{REF}(A) \leq m$

$\text{Rank}(A) =$ number of leading ones in $\text{RREF}(A) \leq n$.

$$\underbrace{\text{Rank}(A)}_{\text{number of leading variables}} + \underbrace{\text{Nullity}}_{\text{number of free variables}} = n$$

$*$ $A_{m \times n} x = b$

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$$

Th: If $A_{m \times n}$, then $Ax = b$ is consistent, iff $b \in CS(A)$

Th: $Ax = 0$ has only the trivial solution, iff column vectors of A are L.I (iff $\text{Rank}(A) = n$).

$$CS(A) = \text{span}(a_1, a_2, \dots, a_n)$$

$$CS(A) = \{ \gamma_1 a_1 + \gamma_2 a_2 + \dots + \gamma_n a_n \}$$

$$\dim(CS(A)) = \text{Rank}(A)$$

Th: $A_{m \times n}$, $Ax = b$ is consistant for every $b \in \mathbb{R}^m$, iff the column vectors of A span $\mathbb{R}^m$.

$$A_{m \times n} = (a_1, a_2, \dots, a_n) \ ; \ a_j \in \mathbb{R}^m$$

$b \in \mathbb{R}^m \Rightarrow b \in CS(A)$

$\Rightarrow b = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$

$b = A\alpha$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

* Result: If $A\alpha = b$ is consistent for every $b \in \mathbb{R}^m$, then $n \geq m$.

* If $A_{m \times n} = (a_1, a_2, \dots, a_n)$; $a_j \in \mathbb{R}^m$ and suppose that $a_1, a_2, \dots, a_n$ are L.I, then $n \leq m$.

So, if the column vectors of $A_{m \times n}$ form a basis for $\mathbb{R}^m$, then $n = m$.

Th: $A_{n \times n}$ is non-singular iff column vectors of A form a basis for $\mathbb{R}^m$ (iff $\mathrm{Rank}(A) = n$).

Th: $A_{m \times n}$, $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$, iff the column vectors of A are L.I.

proof: 1) Assume $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$, we need to show that the column vectors are L.I.

take $b = 0$, $Ax = 0$ has only the trivial solution

$\Rightarrow x_1 a_1 + x_2 a_2 + \dots + x_n a_n = 0$

$\Rightarrow a_1, a_2, \dots, a_n$ are L.I.

2) Assume that the column vectors are L.I, we need to show that $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$:

since $a_1, a_2, \dots, a_n$ are L.I, then
$c_1 a_1 + c_2 a_2 + \dots + c_n a_n = 0$
has only the trivial solution.

So, $Ax = 0$ has only the trivial solution.

Now, for $Ax = b$, assume z & y are two solutions, then:

$Az = b$ & $Ay = b$

$Az - Ay = 0$
$A(z - y) = 0$

$\Rightarrow (z - y)$ is a solution for $Ax = 0$, that is

$z - y = 0 \Longrightarrow z = y.$

So, $Ax = b$ has at most one solution.

note: Rank $(A) = 0$, iff A is the zero matrix.

ex: let $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$, then:

1) Find a basis for the null space of A.
2) Find a basis for the row space of A.
3) Find a basis for the column space of A.
4) What is the rank and nullity of A?

1) $\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Now, solve the system $A\mathbf{x} = \mathbf{0}$

x_1, x_2: leading variables
x_3, x_4, x_5: free variables

So, $x_1 = -x_3 - 2x_4 - x_5 = -s - 2t - w$; $s, t, w \in \mathbb{R}$
$x_2 = -x_3 - x_4 - 2x_5 = -s - t - 2w$

$$\Longrightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-2t-w \\ -s-t-2w \\ s \\ t \\ w \end{pmatrix}$$

$$N(A) = \left\{ \begin{pmatrix} -s-2t-w \\ -s-t-2w \\ s \\ t \\ w \end{pmatrix} ,\ s, t, w \in \mathbb{R} \right\}$$

$$x = s \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + t \underbrace{\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + w \underbrace{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$$

So, a basis for $N(A)$ is $\{v_1, v_2, v_3\}$.

2) Basis for $RS(A)$ is $\{(1,4,5,6,9), (0,1,1,1,2)\}$

3) Because the first & second columns of $REF(A)$ have the leading ones, then the first and second columns of A form a basis for $CS(A)$.

So, A basis for $CS(A)$ is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix} \right\}$

4) $\text{Rank}(A) = \dim(CS(A)) = \dim(RS(A)) = 2$, Nullity $(A) =$