Chapter 2 : Determinants

(2.1 & 2.2 & 2.3) Determinant Properties & Applications

For any square $(n*n)$ matrix A, we can assign a number (scalar) called the determinant of A, denoted by $\det(A)$ or $\det A$ or $|A|$.

We use $|A|$ to decide whether A is singular or non-singular.

How?

Th: A_{n*n} is non-singular, iff $|A| \neq 0$.

Th: A_{n*n} is singular, iff $|A| = 0$.

How to find A?

1) For $(1*1)$ matrix $A = [a_{11}]$:

$$|A| = a_{11}$$

2) For $(2*2)$ matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = a_{11}.a_{22} - a_{12}a_{21}$$

ex: $A = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$, find $|A|$.

$$|A| = (1)(-3) - (0)(4)$$
$$|A| = -3 \neq 0$$

$\Longrightarrow$ A is non-singular $\Longrightarrow$ has inverse

Cofactors & Minors :

Def: let A be (n*n) matrix, let M_{ij} be the (n-1)*(n-1) matrix resulted from deleting the row and the column that contain a_{ij}, then:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{(3*3)}$$

Minor of a_{ij} = det (M_{ij})

Cofactor of a_{ij} = $(-1)^{i+j}$ det (M_{ij})

ex: $A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & 4 & 1 \\ 3 & 6 & -2 \end{bmatrix}$

$M_{11} = \begin{bmatrix} 4 & 1 \\ 6 & -2 \end{bmatrix} \Longrightarrow$ minor of a_{11} = det M_{11} = $-8-6=-14$

$A_{11} = (-1)^{1+1}$ det M_{11} = -14

$M_{12} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \Longrightarrow$ minor of a_{12} = det M_{12} = $-4-3=-7$

$A_{12} = (-1)^{1+2}$ det M_{12} = 7

* A_{n*n} matrix :

det A = $a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$, $i = 1, 2, 3, \dots n$

(Cofactor expansion of det A by ith row)

ex: $i=1 \Longrightarrow$ det A = $a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n}$

det A = $a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$, $j = 1, 2, 3, \dots n$

(Cofactor expansion of det A by jth row) ex: $j=2 \Longrightarrow \det A = a_{12}A_{12} + a_{22}A_{22} + \dots + a_{n2}A_{n2}$

ex: $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 4 & 1 \\ 3 & 5 & -1 \end{bmatrix}$, find det A.

taking the first row $(i=1)$:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \qquad \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} \boxed{1 \quad 0 \quad -2} \\ 2 \quad 4 \quad 1 \\ 3 \quad 5 \quad -1 \end{bmatrix} \end{array}$$

$$= (1)\begin{vmatrix} 4 & 1 \\ 5 & -1 \end{vmatrix} - (0)\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= -9 - 0 + 4$$

$\det A = -5$

taking the second row $(i=2)$:

$$\det A = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \qquad \begin{bmatrix} \underset{-}{1} & \underset{+}{0} & \underset{-}{-2} \\ \boxed{2 \quad 4 \quad 1} \\ 3 & 5 & -1 \end{bmatrix}$$

$$= -(2)\begin{vmatrix} 0 & -2 \\ 5 & -1 \end{vmatrix} + (4)\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix}$$

$$= -20 + 20 - 5$$

$\det A = -5$

* Properties of Determinants:

1) If a row or column of A consists of zeros, then $|A| = 0$.

2) If A has two identical rows or columns, then $|A| = 0$.

ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 3 & 4 \end{bmatrix} \Rightarrow |A| = 0$, since C_1 and C_2 are identical.

3) If a row or a column of A is a multiple of another one, then $|A| = 0$.

ex1: $A = \begin{bmatrix} a & b \\ -3a & -3b \end{bmatrix} \Rightarrow |A| = 0$, because $R_2 = -3R_1$.

ex2: $A = \begin{bmatrix} a & -a \\ b & -b \end{bmatrix} \Rightarrow |A| = 0$, because $C_2 = -C_1$.

4) The determinant of any triangular matrix (upper Δ or lower Δ) or diagonal 主 is:

$|A| = a_{11}\, a_{22}\, a_{33} \dots a_{nn}$

ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -2 & 0 \\ 8 & 6 & 3 \end{bmatrix} \Rightarrow |A| = (1)(-2)(3) = -6$
(since it is lower Δ)

5) $|I| = 1$.

6) $|AB| = |BA| = |A|\,|B| = |B|\,|A|$

7) $|A+B| \neq |A| + |B|$

ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = B$

$$\left.\begin{aligned} &|A| = 1 ,\ |B| = 1 \Longrightarrow |A| + |B| = 2 \\ &A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow |A+B| = 0 \end{aligned}\right\} \ 2 \neq 0$$

8) $|A^K| = (|A|)^K$

proof: $|A^K| = |AAA\ldots A|$ (K times)

$= |A|\,|A|\,|A|\ldots|A|$

$= (|A|)^K$

9) $|\alpha A| = \alpha^n |A|$, α is scalar , A is $(n*n)$ matrix.

ex: $|A| = 5$, find $|2A|$, $|-A|$, A is $(3*3)$ matrix

$|2A| = 2^3 |A| = 8|A|$

$|A|$ [illegible] $^3|A|$ [illegible]

10) $|A^T| = |A|$

Q : Prove that if $A_{n\times n}$ is non-singular, then $|A^{-1}| = \frac{1}{|A|}$

proof: $A A^{-1} = I$

$|A|\,|A^{-1}| = |I|$

$|A|\,|A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

* Determinants of Elementary Matrices:

1) If E is of type I, then $\det E = -1$.

2) If E is of type II, then $\det E = \alpha$ (α is the constant that has multipled by a row of I).

3) If E is of type III, then $\det E = 1$.

ex: 1) $E^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |E| = -1$.

2) $E^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |E| = -4$

3) $E^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |E| = 1$

5 :

1) If we interchange two rows (or two column) of A then $|A|$ changes to $-|A|$

row operation

proof: $|E^{(1)}A| = |E||A| = -|A|$

ex: $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $|A| = \frac{1}{2}$

find $\begin{vmatrix} b & d \\ a & c \end{vmatrix}$, $\begin{vmatrix} c & a \\ d & b \end{vmatrix}$.

$\begin{vmatrix} b & d \\ a & c \end{vmatrix} = -|A| = -\frac{1}{2}$

$|AE^{(1)}| = \begin{vmatrix} c & a \\ d & b \end{vmatrix} = -|A| = -\frac{1}{2}$

column operation

2) If we multiply a row (or a column) by a scalar α, then $|A|$ change to $\alpha|A|$.

proof: $|E^{(2)}A| = |E||A| = \alpha|A|$.

ex: $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$, $|A| = -3$

find $\begin{vmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{vmatrix}$, $\begin{vmatrix} 2a & 2d & 2g \\ \frac{1}{3}b & \frac{1}{3}e & \frac{1}{3}h \\ -3c & -3f & -3i \end{vmatrix}$.

$$\begin{vmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{vmatrix} = 2|A| = -6$$

$$\begin{vmatrix} 2a & 2d & 2g \\ \frac{1}{2}b & \frac{1}{2}e & \frac{1}{2}h \\ -3c & -3f & -3i \end{vmatrix} = (2)(\tfrac{1}{2})(-3)\,|A| = -3|A| = 9$$

3) If we ~~replace~~ replace a row (or a column) by its addition with a multiple of another one, then $|A|$ does not change

proof: $|E^{(3)}A| = |E|\,|A| = |A|$

ex: $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$, $|A| = 10$

find $\begin{vmatrix} a+b & d+e & g+h \\ b & e & h \\ c & f & i \end{vmatrix}$.

$$\begin{vmatrix} a+b & d+e & g+h \\ b & e & h \\ c & f & i \end{vmatrix} = |A| = 10$$

cofactor expansion بدل طريقة

So, we can use Gaussian Elimination to find $|A|$ by transforming the matrix to upper triangular which is easier to find its determinant.

ex 1: Find $\begin{vmatrix} -2 & 1 & 3 \\ 1 & 4 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$$\begin{vmatrix} -2 & 1 & 3 \\ 1 & 4 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 4 & 6 \\ -2 & 1 & 3 \\ 7 & 8 & 9 \end{vmatrix} \quad (R_1 \leftrightarrow R_2)$$

$$= - \begin{vmatrix} 1 & 4 & 6 \\ 0 & 9 & 15 \\ 0 & -20 & -33 \end{vmatrix} \quad \begin{matrix} (R_2 + 2R_1) \\ (R_3 - 7R_1) \end{matrix}$$

$$= - \begin{vmatrix} 1 & 4 & 6 \\ 0 & 9 & 15 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} \quad (R_3 + \frac{20}{9} R_2)$$

$$= -(1)(9)(\tfrac{1}{3})$$

$$= -3$$

note: the minus sign is because of interchanging two rows.

ex2: Find $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 7 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 1 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 7 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{vmatrix} \quad \begin{matrix} \\ (R_2 - R_1) \\ (R_3 - 2R_1) \\ \\ \end{matrix}$$

R_3 is all zeros, so the determinant $= 0$.

* If A is $(n \times n)$ matrix, then:

row operations on A to make ...

$$E_K \cdots E_2 E_1 A = R \quad (\text{the REF of } A)$$

$$|E_K \cdots E_2 E_1 A| = |R|$$

$$|E_K| \cdots |E_2|\,|E_1|\,|A| = |R|$$

But $|E_i| \neq 0$, for all i

So, $|A| = 0$, iff $|R| = 0$.

Q1: If A is $(n \times n)$ matrix, α is scalar, then prove that $|\alpha A| = \alpha^n |A|$.

proof: let
$$A_{n \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & & \vdots \\ \alpha a_{n1} & \cdots & \alpha a_{nn} \end{bmatrix}$$

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$$|\alpha A| = \underbrace{\alpha\,\alpha\,\alpha\,\alpha\,\alpha}_{n \text{ times}} \begin{vmatrix} a_{11} & a_{1n} \\ \vdots & \vdots \\ a_{n1} \cdots & a_{nn} \end{vmatrix} \quad (\text{taking } \alpha \text{ as a common factor from each row})$$

$$\Longrightarrow |\alpha A| = \alpha^n |A|$$

another way to prove that:

$$A = IA$$

$$\alpha A = \alpha I A$$

$$|\alpha A| = |\alpha I A|$$

$$|\alpha A| = |\alpha I|\,|A|$$

But αI is a diagonal matrix, so $|\alpha I| = \underbrace{\alpha\alpha\ldots\alpha}_{n \text{ times}}$

So $|\alpha I| = \alpha^n$

$$\Rightarrow |\alpha A| = \alpha^n |A|$$

Q2: If $A^T A = I$, then show that $|A| = \mp 1$

proof: $|AA^T| = |I|$

$$|A|\,|A^T| = 1$$

$$|A|\,|A| = 1 \qquad (\text{since } |A| = |A^T|)$$

$$|A|^2 = 1$$

$$\Rightarrow |A| = \mp 1$$

Q3: If $AB = I$, then show that $BA = I$.

proof: $AB = I$

$|AB| = |I|$

$|A||B| = 1$

$\Longrightarrow |A| \neq 0$ and $|B| \neq 0$.

$\Longrightarrow$ A and B are non-singular.

$\Longrightarrow$ A^{-1} and B^{-1} are exist.

$B = IB$

$B = (A^{-1}A)B$

$B = A^{-1}(AB)$

$B = A^{-1} I$

$\Longrightarrow \boxed{B = A^{-1}}$

now, $BA = A^{-1}A$

$\Longrightarrow BA = I$

*Applications of Determinants:

1) Adjoint & Inverse.
2) Cramer's Rule.

1) Adjoint & Inverse:

Recall: from elementary schools, if:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Now, if A is $(n \times n)$ matrix, then the adjoint of A is denoted by adj A and it is also an $(n \times n)$ matrix, such that:

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

where $A_{ij} = (-1)^{i+j} \det M_{ij}$

Now, $A\,(\text{adj } A) = \det A \; I$.

If $\det A \neq 0$, then $A\left(\frac{\text{adj } A}{|A|}\right) = I$

$$\Longrightarrow \boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$$

But if $\det A = 0$; then $A(\text{adj} A) = 0$

ex: let $A = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

a) find $\text{adj} A$ b) find A^{-1} (if exists)

a) $\text{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 13 & 9 & -14 \\ 8 & -6 & -4 \\ -12 & -6 & 6 \end{bmatrix}^T$$

$$\text{adj} A = \begin{bmatrix} 13 & 8 & -12 \\ 9 & -6 & -6 \\ -14 & -4 & 6 \end{bmatrix}$$

b) $|A| = 2(13) + 0(9) + 4(-14)$

$= 26 - 56$

$= -30$

$|A| \neq 0 \Longrightarrow A$ is non-singular $\Longrightarrow A^{-1}$ exists

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$$= \frac{1}{-30}\begin{bmatrix} 13 & 8 & -12 \\ 9 & -6 & -6 \\ -14 & -4 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-13}{30} & \frac{-8}{30} & \frac{12}{30} \\ \frac{-9}{30} & \frac{6}{30} & \frac{6}{30} \\ \frac{14}{30} & \frac{4}{30} & \frac{-6}{30} \end{bmatrix}$$

Note: The (i,j) entry of A^{-1} is $\dfrac{A_{ji}}{|A|}$.

ex: $A = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 1 \\ 4 & 2 & 5 \end{bmatrix}$, $|A| = -30$

Find $(3,1)$ entry of A^{-1}.

$$a_{31} \text{ in } A^{-1} = \frac{A_{13}}{|A|} = \frac{\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix}}{-30} = \frac{-14}{-30} = \frac{14}{30}$$

(The same result in the previous example)

2) Cramer's Rule:

A new technique to solve (n×n) linear system $Ax = b$ when A is non-singular using determinants.

$$x_j = \frac{|A_j|}{|A|} \quad , \quad j = 1, 2, 3, \ldots n$$

where A_j is the $(n \times n)$ matrix obtained from A by replacing the jth column of A by b.

ex: Use Cramer's Rule to solve:

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 9$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} , \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , \quad b = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$$|A| = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$$

$$= (1) \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - (2) \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (1) \begin{vmatrix} 2 & 2 \\ & \end{vmatrix}$$

$$= 4 - 10 + 2$$

$$|A| = -4 \neq 0$$

$\Longrightarrow$ A is non-singular

$$|A_1| = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-4}{-4} = 1$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-4} = 2$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Q1: Show that $|AB| = |A||B|$.

proof: Take two cases:

case 1: B is singular $\Longrightarrow$ AB is singular.

B is singular $\Longrightarrow |B| = 0 \Longrightarrow |A|\,|B| = 0$
AB is singular $\Longrightarrow |AB| = 0$

So, $|AB| = |A||B|$

case 2: B is non-singular $\Longrightarrow$ B is row equivalent to I

$\Longrightarrow B = E_K E_{K-1} \dots E_2 E_1 I$
$\Longrightarrow B = E_K E_{K-1} \dots E_2 E_1$

$$AB = A\, E_K E_{K-1} \dots E_2 E_1$$

$$|AB| = |A|\, |E_K E_{K-1} \dots E_2 E_1|$$

$$\Longrightarrow |AB| = |A|\,|B|$$

note: Any non-singular matrix can be written as a product of elementary matrices.

Q2: Show that $A_{n \times n}$ is singular, iff $|A| = 0$.

$R = E_K E_{K-1} \dots E_2 E_1 A$ (where R is the REF of A)

$|R| = |E_K| |E_{K-1}| \dots |E_2| |E_1| |A|$

But, $|E_i| \neq 0$, for all i.

So, $|R| = 0$, iff $|A| = 0$

Now, A is singular, iff R contains a row consisting entirely of zeros.

And, R contains a row consisting entirely of zeros, iff $|R| = 0$.

But, $|R| = 0$, iff $|A| = 0$.

So, $A_{n \times n}$ is singular, iff $|A| = 0$.