6.1 Eigenvalues and Eigenvectors

Def Let $A \in \mathbb{R}^{n \times n}$. A scalar λ is called an eigenvalue of A if $Ax = \lambda x$ for some nonzero vector x. The vector x is called an eigenvector belonging to λ.

Def Let $A \in \mathbb{R}^{n \times n}$ and λ an eigenvalue of A. The eigenspace corresponding to the eigenvalue λ is the set of all eigenvectors belonging to λ in addition to the zero vector.

Note According to the above definition:

Eigenspace corresponding to the eigenvalue λ

$$= N(A - \lambda I) = \{ x : (A - \lambda I)x = 0 \}$$

$$= \{ x : Ax = \lambda x \}$$

$$= \{ x : x \text{ eigenvector belonging to } \lambda \} \cup \{ \text{zero vector} \}$$

Note

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow (A - I\lambda)x = 0 \text{ has nontrivial solution}$$
$$\Leftrightarrow N(A - \lambda I) \neq \{0\}$$
$$\Leftrightarrow (A - \lambda I) \text{ is singular}$$
$$\Leftrightarrow \det(A - \lambda I) = 0$$

Def The polynomial defined by

$$p(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial.

Note To find the eigenvalues of A, solve $p(\lambda) = 0$.

Note $\det(A) = \prod_{j=1}^{n} \lambda_j = \lambda_1 \cdot \lambda_2 \cdot \ldots \lambda_n$

Note $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{j=1}^{n} \lambda_j$

Note To find the eigenvectors belonging to the eigenvalue λ use row operations to find the RREF of

$$\left[A - \lambda I \mid 0 \right].$$

Example $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

1) Find the eigenvalues of A.
2) Find the eigenspace corresponding to each eigenvalue
3) Verify that $\det(A) = \prod_j \lambda_j$
4) Verify that $tr(A) = \sum_j \lambda_j$

Solution

1) $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2$

$$= 4 - 5\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3$$

2) $\lambda_1 = 2 \Rightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Eigenspace corresponding to $\lambda_1 = 2$ is $\left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\lambda_2 = 3 \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Eigenspace corresponding to $\lambda_2 = 3$ is $\left\{ \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

3) $$\left.\begin{array}{l} \det(A) = \begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} = 4 + 2 = 6 \\ \prod_{j=1}^{2} \lambda_j = \lambda_1 \cdot \lambda_2 = (2)(3) = 6 \end{array}\right\} \Rightarrow \det(A) = \prod_j \lambda_j$$

4) $$\left.\begin{array}{l} tr(A) = \sum_{i=1}^{2} a_{ii} = a_{11} + a_{22} = 4 + 1 = 5 \\ \sum_{j=1}^{2} \lambda_j = \lambda_1 + \lambda_2 = 2 + 3 = 5 \end{array}\right\} \Rightarrow tr(A) = \sum_j \lambda_j$$