## 1.2 Row Echelon Form (REF).

Df. An $m \times n$ matrix is said to be in REF if and only if:

(1) the first nonzero entry in each nonzero row is 1 called the leading one or the pivot 1.

(2) the leading 1 in the $k^{th}$ row is to the right of the leading 1 in the $(k-1)$ row

(3) Zero rows are below the nonzero rows.

Ex. $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is in REF

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not in REF

$C = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is not in REF

$$D = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in REF.}$$

$$E = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \text{ is in REF.}$$

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is in REF.}$$

Rmk: (1) Any matrix can be written in REF using the row operations.

(2) the process of using row operations I, II, and III to transform a linear system into one whose augmented matrix is in REF is called Gaussian Elimination Method.

**Ex.** Use Gauss Elimination method to solve the following systems.

① $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \\ -2x_1 + 2x_2 = -2 \end{cases}$

**Sol.** The augmented matrix is $[A : b]$, i.e.,

$$\begin{array}{cc} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -2 & 2 & -2 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$\xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{-4R_2+R_3} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

The equivalent system is

$$x_1 + x_2 = 1$$
$$x_2 = -1$$
$$0 = 1 \quad \text{(impossible)}$$

$\Rightarrow$ the system is inconsistent (No solution).

(2) $$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 - x_2 + x_3 = 2 \\ 4x_1 + 3x_2 + 3x_3 = 4 \\ 3x_1 + x_2 + 2x_3 = 3 \end{cases}$$

Sol. $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow[\substack{-2R_1+R_2 \\ -4R_1+R_3 \\ -3R_1+R_4}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{5R_2+R_3 \\ 5R_2+R_4}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equivalent system is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + \tfrac{1}{5}x_3 &= 0 \end{aligned}$$

$x_1, x_2$ leading

$x_3$ free

Let $\boxed{x_3 = t}$

$\Rightarrow \boxed{x_2 = -\frac{1}{5}x_3 = -\frac{1}{5}t}$

$x_1 + 2(-\frac{1}{5}t) + t = 1 \Rightarrow \boxed{x_1 = 1 - \frac{3}{5}t}$

$\therefore$ the solution set is

$$\left\{ \left(1-\tfrac{3}{5}t, -\tfrac{1}{5}t, t\right) : t \in \mathbb{R} \right\}.$$

③ $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$

Sol. the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent (has no solution).

④ $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$

Sol. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$

$$\xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The equivalent system is

$$x_1 + x_2 + x_3 = 0 \qquad x_1, x_2 \text{ leading}$$
$$x_2 + x_3 = 0 \qquad x_3 \text{ free}$$

Let $\boxed{x_3 = r} \Rightarrow x_2 = -x_3 = -r$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 - r + r = 0$

$\boxed{x_1 = 0}$

$\therefore$ the solution set $= \{(0, -r, r) : r \in \mathbb{R}\}$.

## Reduced Row Echelon Form (RREF).

**Df.** An $m \times n$ matrix is said to be in RREF if:

(1) The matrix is in REF.

(2) The first nonzero entry in each row (1's) is the only nonzero entry in its column.

Examples.

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ is in RREF.

$$B = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in RREF.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is NOT in RREF.

$$D = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$ is NOT in RREF.

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in RREF

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is not in RREF.

# Gauss-Jordan Elimination Method

is the process of using elementary row operations on the augmented matrix $[A:b]$ of the the system $Ax=b$ to transform it into a system in RREF.

Ex. Use Gauss-Jordan reduction to solve

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Sol. the augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-3R_1+R_2 \\ -2R_1+R_3}]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$x_1, x_2, x_3$ are leading variables

$x_4$ free variable.

let $\boxed{x_4 = t}$

$R_1: \quad x_1 - x_4 = 0 \Rightarrow \boxed{x_1 = t}$

$R_2: \quad x_2 + x_4 = 0 \Rightarrow \boxed{x_2 = -t}$

$R_3: \quad x_3 - x_4 = 0 \Rightarrow \boxed{x_3 = t}$

$\Rightarrow$ the solution set is

$\{(t, -t, t, t) : t \in \mathbb{R}\}$

## Overdetermined and underdetermined systems

Df: An $m \times n$ linear system is called underdetermined system if $m < n$, and it is called overdetermined if $m > n$.

Ex ① page 17 is overdetermined system.

Ex ② page 18 " " " .

Ex ③ page 20 " underdetermined. system .

Ex ④ page 20 " " " .

Rmk: (i) An underdetermined linear system always has a free variable. So it is either inconsistent or it has infinite solutions. It is not possible to have aunique solution.

(ii) An overdetermined linear system Can not tell ( All cases possible).

## Homogeneous systems

<u>Df.</u> An $m \times n$ lin. system is called <u>homogeneous</u> if all right hand of every equation is zero. that is

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0
\end{aligned}$$

(i.e $b_1 = b_2 = \cdots = b_n = 0$).

<u>Rmk.</u> (1) A homogeneous lin. system is always consistent since $x_1 = x_2 = \cdots = x_n = 0$ is a solution called the <u>zero solution</u> or the <u>trivial solution</u>

(2) A homog. system is either has a unique solution ($x_1 = x_2 = \cdots = x_n = 0$) if it has no free variables or it has infinite solutions if it has a free variable.

(3) An underdetermined homog. linear system always has infinite solutions.

<u>Example.</u> See Ex. page 23.

<u>Ex.</u> Consider the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ -x_1 + x_2 + \beta x_3 = 0 \end{cases}$$

(a) Is it possible for the system to be inconsistent? Explain.

(b) for what values of $\beta$ will the system have infinitely many solution?

<u>Ans.</u> (a) No, since $x_1 = x_2 = x_3 = 0$ is a solution.

(b) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & \beta+1 & 0 \end{array}\right]$$

$$\xrightarrow[-3R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \beta-2 & 0 \end{array}\right]$$

If $\beta \neq 2$, the system has unique solution $(x_1 = x_2 = x_3 = 0)$

If $\beta = 2$, " " " infinite solution.

Ex. Consider the linear system

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + \alpha x_3 = \beta. \end{cases}$$

For what values of $\alpha$ and $\beta$ does the system have:

(1) unique solution

(2) no solution

(3) infinitely many solutions.

Sol. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & \alpha & \beta \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

(1) Unique solution if $\alpha \neq 1$, $\beta \in \mathbb{R}$.

(2) No solution if $\alpha = 1$, $\beta \neq 2$

(3) Infinitely many solution if $\alpha = 1$, $\beta = 2$