Math 234:- Introduction to linear Algebra

Ch [I]: Matrices and systems of equations:-

1.1) System of linear equations:-

Def:- a linear equation in (n) unknowns is an equation of the form :-

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where $a_1, a_2, \dots a_n$, b are constants (real num) and $x_1, x_2 \dots x_n$ are variables.

E.g $ax + by = c$ is a linear equation of two variables (x & y).
a, b, c are constants.

Def: a linear system of (m) equations and (n) unknowns is a system of the form:-

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}\right\} \textcircled{*}$$

where a_{ij}'s and b_{ij}'s are real numbers

m: # of equations
n: # of variables / unknowns.

This system is called $m \times n$ system

E.g ① $2x_1 - x_2 = 5$ is 2×2 linear system
$x_1 + 3x_2 = 6$ (معادلتين بمجهولين)

E.g ② $x_1 - x_2 + x_3 = 2$ is 2×3 linear system
$2x_1 - x_2 + x_3 = 7$

E.g ③ $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \\ x_1 = 4 \end{cases}$ is 3×2 linear system.

Remark:-

- By a solution of $m \times n$ system ⊛ we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots x_n)$ that satisfies all the equations of the system ⊛

E.g $(3,1)$ is a solution of the system in E.g①
(2-tuple) $x_1 = 3$, $x_2 = 1$
since, $2(3) - (1) = 5$ ✓
$3 + 3(7) = 6$ ✓

E.g Solve the following systems:-

(1) $\begin{cases} x_1 + x_2 = 4 \\ x_1 - x_2 = 2 \end{cases}$

Add:

$2x_1 = 6 \rightarrow \boxed{x_1 = 3}$ & $\boxed{x_2 = 1}$

2-tuple:- $(3,1)$ is a solution (unique solution).

(2) $x_1 + 2x_2 = 4$

$x_1 + 2x_2 = -2$

Eq① - Eq② $\Rightarrow$ $0 = 6$ contradiction equation (false statement)

Impossible :- The system has no solution. (they are parallel, same slope).

E.g $2x_1 - x_2 = 3$

$-4x_1 + 2x_2 = -6$

$4x_1 - 2x_2 = 6$

$-4x_1 + 2x_2 = -6$

Add $0 = 0$ true statement (everywhere)

has infinitely many solutions.

$2x_1 - x_2 = 3 \rightarrow x_2 = 2x_1 - 3$

let $\boxed{x_1 = t}$ $\boxed{x_2 = 2t - 3}$

Solution set $= \{(x_1, x_2) = (t, 2t-3) : t \in \mathbb{R}\}$

OR $2x_1 = x_2 + 3 \rightarrow x_1 = \frac{x_2 + 3}{2}$

let $x_2 = \alpha$ $x_1 = \frac{\alpha + 3}{2}$

Soln set $= \{(\frac{\alpha+3}{2}, \alpha) : \alpha \in \mathbb{R}\}$

- **Remark:-**

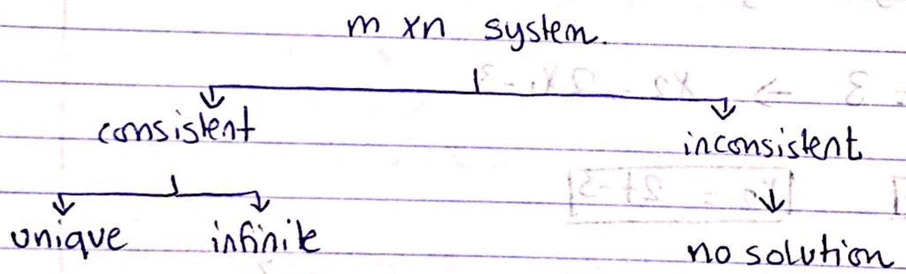
In general, there are three possibilities for 2x2 System :-

(1) unique solution (the lines intersect at a point)
(2) No solution (the lines are parallel)
(3) Infinite solutions (Both eqs. represent the same line).

In general, any mxn system has one of the following possibilities:-

(1) has a unique solution (consistent)
(2) has infinitely many solutions. (consistent)
(3) has no solution. (inconsistent)

That is :-



Def:- two systems of eqs involving the same variables are said to be equivilant. if:-
(1) they have the same solution set.
(2) same # of variables.

E.g verify that the following systems are equivivilant

a) $x_1 + x_2 = 2$
$x_1 - x_2 = 6$

b) $x_1 + x_2 = 2$
$2x_2 = -4$

Sdn:- Yes, since:-
(1) same # of variables $(x_1 + x_2)$.
(2) a & b have the same solution set $(4, -2)$

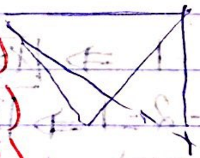
- $n \times n$ systems (square systems).

Def:- a system is said to be strict triangular form if:- (should be square)

in the k^{th} eq. the coefficients of the first $(k-1)$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, 2, 3, \ldots, n)$

E.g: the following system is in strict triangular form

$$\begin{cases} x_1 + x_2 + x_3 = 8 & (1,1,1) \\ 0x_1 + 2x_2 + x_3 = 5 & (0,2,1) \\ 0x_1 + 0x_2 + 3x_3 = 9 & (0,0,3) \end{cases}$$



But
$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 2x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 9 \end{aligned}\right\} \text{non strict triangular.}$$

E.g
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 6 \\ x_2 + x_3 \boxed{-x_4} &= 0 \\ x_3 + x_4 &= 1 \\ 2x_4 &= 6 \end{aligned}$$

→ if this was removed it's also strict triangular we only look for the previous.

$(1,1,1,1)$
$(0,1,-1,0)$
$(0,0,1,1)$
$(0,0,0,2)$

⇒ in strict triangular form.

*Remark:- (1) The system in strict triangular form is easy to solve.
- → back substitution
- ↳ forward substitution

(2) The method that we use to solve such a system is called back-substitution.

E.g Use back substitution to solve the following system:-

$$x_1 + 2x_2 + 2x_3 + x_4 = 5 \quad ①$$
$$3x_2 + x_3 - 2x_4 = 1 \quad ②$$
$$-x_3 + 2x_4 = -1 \quad ③$$
$$4x_4 = 4 \quad ④$$

Solution:- Notice that the system is in strict triangular form.

Eq ④ $\Rightarrow \boxed{x_4 = 1}$

Eq ③ $\Rightarrow -x_3 + 2 = -1 \Rightarrow \boxed{x_3 = 3}$

Eq ② $\Rightarrow 3x_2 + 3 - 2 = 1 \Rightarrow \boxed{x_2 = 0}$

Eq ① $\Rightarrow x_1 + (0) + 2(3) + 1 = 5 \Rightarrow \boxed{x_1 = -2}$

∴ solution set $(-2, 0, 3, 1)$ (unique)

Question:- How to transform a system in strict triangular form?

Ans:- we use the elementary row operations:-

(I) Interchange two rows.
(II) Multiply a row by a non-zero constant
(III) Replace a row its sum with a multiple of a non-zero row.

Ex: Convert the following system into a strict triangular form & then solve it.

$$-x_2 - x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 6$$
$$2x_1 + 4x_2 + x_3 - 2x_4 = -1$$
$$3x_1 + x_2 - 2x_3 + 2x_4 = 3$$

Soln:- Augment Matrix

$$\left[\begin{array}{cccc:c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} \textcircled{1} & 1 & 1 & 1 & 6 \\ \textcircled{0} & \textcircled{-1} & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{bmatrix} \begin{array}{l} \\ \\ -2R_1 + R_3 \\ -3R_1 + R_4 \end{array} \longrightarrow$$

$$\left[\begin{array}{cccc|c} \enclose{circle}{1} & 1 & 1 & 1 & 6 \\ 0 & \enclose{circle}{-1} & -1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{-3} & -2 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

مع مقابلها بإشارة
عكسية فا[illegible]
الصفر
$2R_2 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right] \quad -2R_2 + R_4$$

$$\left[\begin{array}{cccc|c} \enclose{circle}{1} & 1 & 1 & 1 & 6 \\ 0 & \enclose{circle}{-1} & -1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{-3} & -2 & -13 \\ 0 & 0 & 0 & \enclose{circle}{-1} & -2 \end{array}\right] \quad -2R_2 + R_4$$

are called pivot.

System becomes

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 6 \\ -x_2 - x_3 + x_4 &= 0 \\ -3x_3 - 2x_4 &= -13 \\ -x_4 &= -2 \end{aligned}$$

and is in strict triangular form & You can solve it.

Answer:- $(2, -1, 3, 2)$

⚠ not all square systems can be done using this method (one row may totally be zero).

E.g $x_1 + x_2 = 5$
$-x_1 - x_2 = -5$

Remark:- In general, if an $n \times n$ system can be reduced to strictly triang. form then it will have a unique solution

However, this method will fail if at any stage of the process all a pivot column(s) are (0).

1.2 Row Echelon Form (REF)

Def: a matrix is said to be in REF if:-

(1) The 1st non-zero entry in each non-zero row is 1 (leading coefficient).

(2) if a row K doesn't consist entirely of zeros, the # of leading zero entries in row (K+1) is greater than the # of leading zero entries in row K. على عينه

(3) If there are rows whose entries are all zeros, they are below the rows having non-zero entries. بالآخر

Ex:- Which of the following matrices are in REF?

$$A = \begin{array}{c} \begin{bmatrix} \textcircled{1} & 4 & 2 \\ 0 & \textcircled{1} & 5 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

✓ in REF

$$B = \begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

(1) ✓
(2) ✓
(3) X fails

X not in REF

$$C = \begin{bmatrix} \textcircled{2} & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

X not in REF (1) fails.

$$D = \begin{bmatrix} \textcircled{1} & 3 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in REF.

$E = \begin{bmatrix} \textcircled{1} & 4 & 6 \\ 0 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 3 \end{bmatrix}$ X not REF

↓ matrix not system
we can't switch (not augmented matrix).

$F = \begin{bmatrix} \textcircled{1} & 1 & 1 \\ \textcircled{1} & 1 & -1 \\ \textcircled{1} & 1 & 1 \end{bmatrix}$ X not in REF

$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ① ✓ ② ✓ ③ ✓ } if there is non-zero. is in REF.

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* <u>Remark:</u> the process of using row operations I, II, III to transform a linear system to one whose augmented matrix is in REF is called Gaussian Elimination → (REF).

<u>Def:</u> An $m \times n$ system is said to be overdetermined if $m > n$ (# eqs. > # unknowns).

- Overdetermined systems are <u>usually</u> (but not always) inconsistent.

<u>Eg</u> Use Gaussian Elimination to solve

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \\ -2x_1 + 2x_2 = -2 \end{cases}$$

<u>Soln:</u> The augmented matrix:-

① $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -2 & 2 & -2 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \rightarrow$ ② $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 0 \end{array}\right]$ $2R_1 + R_3$

③ $-\frac{1}{2}R_2 \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 0 \end{array}\right]$ ④ $-4R_2 + R_3 \left[\begin{array}{cc|c} 1 & 1 & +1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{array}\right]$ ⑤ $\frac{1}{4}R_3 \left[\begin{array}{cc|c} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & \textcircled{1} \end{array}\right]$ ✓

↓
REF
not STF → not square

$x_1 + x_2 = 1$

$x_2 = -1$

$0x_1 + 0x_2 = 1 \Rightarrow 0 = 1$ (contradiction) impossible

→ The system has no solution (inconsistent).

Ex: Solve by using Gaussian Elimination.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + -x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Soln:- The augmented matrix

①

لازم 1 بالعكس نبدل رووز.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \rightarrow$$

②

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] \begin{array}{l} \\ -2R_1 + R_2 \\ -4R_1 + R_3 \\ -3R_1 + R_4 \end{array}$$

③ $-\frac{1}{5}R_2$

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] -\frac{1}{5}R_2$$

④

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ 5R_2 + R_3 \\ 5R_2 + R_4 \end{array}$$

↓ in REF

$x_1 + 2x_2 + x_3 = 1$ $\quad x_1, x_2$ leading

$x_2 + \frac{1}{5}x_3 = 0$ $\quad x_3$ is free (anything) ⇒ true statement

$\boxed{x_3 = t}$ and back substitution ⇒ $x_2 + \frac{t}{5} = 0$

$\boxed{x_2 = \frac{-t}{5}}$

⚠ leading / free or x_n

$x_1 + 2(-\frac{1}{5}t) + t = 1$ $\quad \boxed{x_1 = 1 - \frac{3}{5}t}$

solution set

$= \{ (x_1, x_2, x_3) = ((1-\frac{3}{5}t), -\frac{1}{5}t, t) : t \in \mathbb{R} \}$

$\Rightarrow$ infinitely many solutions.

Def:- a system $(m \times n)$ is said to be underdetermined system if $m < n$

Rmk:- It's possible for this system to be inconsistent (if consistent $\Rightarrow$ always infinitely many solutions). They are usually consistent with infinitely many solution. It is not possible to have a unique solution. (m leading there are always free).

Ex:- Solve by using Gaussian System

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} ① & 2 & 1 & 1 \\ 0 & 0 & 0 & ① \end{array}\right] \; -2R_1 + R_2$$

$$\Downarrow$$

$$x_1 + 2x_2 + x_3 = 1$$

$$0 = 1 \text{ (impossible)}$$

↳ has no solution.

1.2 (REF)

Ex: Solve $x_1 + x_2 + x_3 = 0$ → underdetermined.

$x_1 - x_2 - x_3 = 0$

Soln:- The augmented matrix:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] -R_1+R_2$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] -\tfrac{1}{2}R_2 \text{ in REF}$$

x_1, x_2 are leading

$x_3 = t$ is free

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ \quad\quad x_2 + x_3 = 0 \end{cases} \begin{array}{l} \rightarrow x_1 + x_2 + x_3 = 0 \rightarrow x_1 + t - t = 0 \; \boxed{x_1 = 0} \\ \rightarrow x_2 = -x_3 \rightarrow \boxed{x_2 = -t} \end{array}$$

* one leading variable is sufficient to have infinitely

Soln Set = $\{(0, -t, t) : t \in \mathbb{R}\}$

Def:- Reduced Row Echelon Form (RREF)

a matrix is said to be in (RREF) if:-

(1) The matrix is in REF.

(2) The 1st non-zero entry in each row (1's) is the only non-zero entry in its column.

E.g which of the following is in RREF?

(1) $A = \begin{bmatrix} \textcircled{1} & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$ (1) ✓ (2) ✓ → in RREF

(2) $B = \begin{bmatrix} 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (1) ✓ (2) ✓ → in RREF

(3) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✗ (1) $\overset{\to (3)X}{X}$ → not in RREF.

(4) $D = \begin{bmatrix} \textcircled{1} & 4 & 6 \\ 0 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 3 \end{bmatrix}$ ✗ → not in RREF.

RMK : The proces of using elementary row operations to transform a system into RREF is called Gauss-Jordan reduction.

Ex:- Use Gauss-Jordan reduction to solve

$$\left.\begin{aligned} -x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}\right\} \text{Under}$$

Soln:- the augmented matrix is

$$\begin{bmatrix} -1 & 1 & 1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} -R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \begin{array}{l} \\ -3R_1+R_2 \\ -2R_1+R_3 \end{array}$$

$$\frac{1}{4}R_2 \begin{bmatrix} -1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{bmatrix} \begin{array}{l} R_1+R_2 \\ R_2 \\ -R_2+R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} -\frac{1}{3}R_3 \rightarrow \left[\begin{array}{cccc|c} \textcircled{1} & 0 & 0 & -1 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -1 & 0 \end{array}\right] \begin{array}{l} R_2+R_3 \\ \text{RREF.} \end{array}$$

$(x_1, x_2, x_3) \rightarrow$ leading

$x_4 = t \rightarrow$ free

$x_1 - x_4 = 0 \quad \boxed{x_1 = t}$

$x_2 + x_4 = 0 \quad \boxed{x_2 = -t}$

$x_3 - x_4 = 0 \quad \boxed{x_3 = t}$

$\boxed{x_4 = t.}$

Soln set $= \{ (t, -t, t, t) : t \in \mathbb{R} \}$.

<u>EX:</u> (Old Exam)

Consider a linear system $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + \alpha x_3 = \beta \end{cases}$

For what values of α and β does the system has ① a unique soln ② no soln ③ inf. many. solns.

<u>Solution</u>:- Augmented Matrix:-

$$\begin{bmatrix} \textcircled{1} & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & \alpha & \beta \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & \alpha-1 & \beta-2 \end{bmatrix} \begin{array}{l} \\ -2R_1+R_2 \\ -R_1+R_3 \end{array}$$

$$\begin{bmatrix} \textcircled{1} & -1 & 1 & 2 \\ 0 & \textcircled{1} & -1 & 1/3 \\ 0 & 0 & \alpha-1 & \beta-2 \end{bmatrix}$$

① If $\alpha = 1$, $\beta = 2 \rightarrow$ infinitely many solns.
$\downarrow$ x_3 free.

② $\alpha = 1$, $\beta \neq 2 \rightarrow$ no soln.

③ $\alpha \neq 1$, $\beta \in \mathbb{R}$ unique soln.

$\frac{y}{x}$

Homogeneous system:-

Def: Homogeneous system /

if the R.H.S of ⊛ in sec 1.1 = 0.

$$\left.\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \rightarrow \text{always consistent}$$

* that is the homog. system is always consistent since
$x_1 = x_2 = \cdots = x_n = 0$ is a solution.
(trivial solution)

RMK:- if $m \times n$ homogeneous system has a unique solution it must be the trivial solution.

Theorem:- an $m \times n$ homogeneous system of linear equations has a non-trivial solution if $m < n$
(under determined sys).

$\boxed{m \leq n \text{ makes it false}}$

E.g 1st Eg in the lecture

$x_1 + x_2 + x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$\boxed{1.2}$ more examples

E.g. Consider

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ -x_1 + x_2 + \beta x_3 = 0 \end{cases}$$

(a) is it possible for the system to be inconsistent?
(b) for what values of β will the system have infinitely many solutions?

Soln:- (a) No. Since the system is homogeneous.

(b) $$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & (1+\beta) & 0 \end{bmatrix} \begin{matrix} \\ -2R_1 + R_2 \\ R_1 + R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \beta - 2 & 0 \end{bmatrix}$$

if $\beta = 2 \to$ we have infinitely many solutions
(we have non-trivial soln).

if $\beta \neq 2 \to$ unique soln which is the trivial soln $(x_1 = x_2 = x_3 = 0)$

(H.W) (old exam).

Suppose that

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 2 & 3 & 1 & 1 & -1 \\ 0 & 1 & 1 & a & b \end{array}\right]$$

is the augmented matrix of some linear system.

Find a & b that make the system.

(1) inconsistent.

$a = -3 \quad b \neq 1$

(2) Has a Unique soln.

~~$a \neq -3$~~ Not Possible

(3) Has infinitely many solns?

$a = -3 \quad b = 1$

$\boxed{1.3}$ Matrix Arithmetic

Def:- Matrix is a rectangular array of m rows and n columns. Thus, if A is $m \times n$ matrix, then A has the form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

- $m \times n$ is the size (order) or the dimension of the array.
- For simplicity, we write

$$A = (a_{ij}) , \Rightarrow \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, \ldots, n \end{matrix}$$

- a_{ij} is the entry in the ith row & the jth column. and it is called the entry.

Ex:- $A = \begin{bmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{bmatrix}_{2 \times 3}$

size $= 2 \times 3$ $\quad a_{23} = 10$ $\quad a_{21} = 6$

* Column vector :- is an $m \times 1$ matrix

Eg $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

* Row vector :- is $1 \times n$ matrix

Eg $B = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}_{1\times 4}$

Ex: The solution of the system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{cases}$$ is $(2,1)$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2, & 1 \end{bmatrix}$

* Euclidean n-space

$\mathbb{R}^m$: All $m \times 1$ matrix of real numbers.

Eg $\mathbb{R}^3 = \left\{ x : x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$

Eg $\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$$x \in \mathbb{R}^m \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m \times 1}$$

$\mathbb{R}^{1\times n}$: All $1\times n$ matrices with real entries.

$$x \in \mathbb{R}^{1\times n} \Rightarrow x = [x_1\ x_2\ \cdots\ x_n]_{1\times n}$$

- $\mathbb{R}^{m\times n}$ All mn matrices.

E.g $\mathbb{R}^{3\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} : \begin{array}{l} a_{ij} \in \mathbb{R} \\ i = 1,2,3 \\ j = 1,2 \end{array} \right.$

If A is $m\times n$ matrix then the row vectors of A are

$$\vec{a_i} = (a_{i1}, a_{i2}, \dots, a_{in}) \,,\quad i = 1, 2, \dots m$$

and the column vectors

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad j = 1, 2, \dots n$$

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \end{bmatrix}_{2\times 3}$

Rows: $\vec{a_1} = (1,2,3)$

$\vec{a_2} = (0,4,-1)$

Columns:- $a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$a_3 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Def:- Two $m \times n$ matrices A and B are equal if:
$a_{ij} = b_{ij}$ for all i,j.

Ex:- let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A \neq B$ since $a_{11} = 1$, $b_{11} = 0$, $a_{11} \neq b_{11}$ enough

note:- they have the same size.

Ex: If $A = \begin{bmatrix} 1 & 3 \\ 2x+1 & 3y^2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

If $A = B$, find x & y.

Soln: $2x+1 = 3 \rightarrow x = 1$
$3y^2 = 9 \rightarrow y = \pm\sqrt{3}$

---

* Scalar Multiplication

Def:- if $A_{m\times n}$ and α is scalar $\rightarrow$ real. $\rightarrow$ complex.

then $\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$

for all i,j.

Ex:- if $A = \begin{bmatrix} 2 & 1 \\ 0 & -5 \end{bmatrix}$ find $4A$

$4A = \begin{bmatrix} 8 & 4 \\ 0 & -20 \end{bmatrix}$

- Matrix addition and subtraction

let $A = (a_{ij})$, $B = (b_{ij})$ are both $\boxed{m \times n}$ matrices

Then $A \pm B = (a_{ij} \pm b_{ij})$, $\forall i, j$. $C_{m \times n}$

Eg $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} -1 & 5 \\ 6 & 7 \end{bmatrix}$ undefined

Find

① $A + C$ is undefined.

② $A + B = \begin{bmatrix} 5 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$

③ $3B - 3A = \begin{bmatrix} -5 & -6 & 5 \\ -12 & -15 & -18 \end{bmatrix}$

1.3 continue

- zero matrix O is a matrix whose entries are all zero.

Eg $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}_{1\times 4}$

- $A + O = A = O + A$ $\quad O_{2\times 4}$

- $A - A = O$

- $A - A = A + (-A)$

  $-A$: is called additive inverse of A.

Eg An additive inverse of

$$A = \begin{bmatrix} -1 & 6 \\ 4 & 3 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & -6 \\ -4 & -3 \end{bmatrix}$$

- Matrix Multiplication and linear systems.

Def:- If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$.

Then $AB = (c_{ij})_{m\times p}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

E.g. $A = \begin{bmatrix} 1 & ③ \\ 6 & -1 \end{bmatrix}_{2\times ②}$ $B = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \end{bmatrix}_{②\times 3}$

$2\times ③$ ②

Find AB

$$AB = \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+3\times0 & 1\times5+3\times1 & 1\times2+3\times-3 \\ 6\times1+-1\times0 & 6\times5+-1\times1 & 6\times2+-1\times-3 \end{bmatrix} = \begin{bmatrix} 1 & 8 & -7 \\ 6 & 29 & 15 \end{bmatrix}_{2\times3}$$

$BA = B_{2\times ③} \cdot A_{②\times 2}$ Undefined.

Rmk:- $AB \neq BA$ (in general).

E.g $A = \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times ②}$ $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{②\times 1}$

$$AB = \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times ②}\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{②\times 1} = \begin{bmatrix} 1\times3+3\times1 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}_{1\times1}$$

↓ every real num can be written as a matrix (1×1) from 1×2·2×1

$$BA = \underset{2\times1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \underset{1\times2}{\begin{bmatrix} 1 & 3 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}}$$

* Linear System

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = b_n$$
$$;$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\in \mathbb{R}^m}$$

A: coefficient matrix

$Ax = b$ $\qquad$ $Ax = 0$

Coefficient Matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \quad \text{(unknowns)}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m \quad \text{(constants; knowns)}$$

Eg Write as a matrix form

$4x_1 + 2x_2 + x_3 = 1$

$5x_1 + 3x_2 + 7x_3 = 2$

$$\begin{bmatrix}4&2&1\\5&3&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

Sol

$$\underbrace{\begin{bmatrix}4&2&1\\5&3&7\end{bmatrix}}_{A\ 2\times\textcircled{3}}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{x\ \textcircled{3}\times 1}=\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{2\times 1}$$

$b\ 2\times 1$

$Ax = b$, where A, x, b as above.

Now, consider the system $\boxed{Ax=b}$ — ⊛

⊛ can be rewritten as

$b = x_1a_1 + x_2a_2 - - - + a_nx_2.$

$b = x_1\textcircled{$a_1$} + x_2a_2 + \cdots + a_nx_n$

$$\begin{bmatrix}1\\2\end{bmatrix} = x_1\begin{pmatrix}4\\5\end{pmatrix} + x_2\begin{pmatrix}2\\3\end{pmatrix} + x_3\begin{pmatrix}1\\7\end{pmatrix}$$

Ex:- In the previous example.

$$b = \begin{bmatrix}1\\2\end{bmatrix} = x_1\underset{a_1}{\begin{pmatrix}4\\5\end{pmatrix}} + x_2\underset{a_2}{\begin{pmatrix}2\\3\end{pmatrix}} + x_3\underset{a_3}{\begin{pmatrix}1\\7\end{pmatrix}}$$

Def:- (Linear Combination)

If $a_1, a_2, \ldots, a_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then the sum column vector $\boxed{c_1a_1 + c_2a_2 + \cdots + c_na_n}$ is said to be a linear combination of $a_1, a_2, \ldots a_n$

E.g is $b = \begin{pmatrix}2\\24\end{pmatrix}$ a linear combination of $a_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $a_2 = \begin{pmatrix}0\\5\end{pmatrix}$??

Ans $\alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\24\end{pmatrix}$

$$b = \begin{pmatrix}2\\24\end{pmatrix} = \textcircled{2}\underset{a_1}{\begin{pmatrix}1\\2\end{pmatrix}} + \textcircled{4}\underset{a_2}{\begin{pmatrix}0\\5\end{pmatrix}}$$

∴ b is a linear combination of a_1 and a_2.

E.g is $b = \begin{pmatrix}1\\1\end{pmatrix}$ a linear combination of $a_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $a_2\begin{pmatrix}2\\4\end{pmatrix}$?

Suppose yes $\Rightarrow \exists\ \alpha_1 \alpha_2$ such that $\alpha_1 a_1 + \alpha_2 a_2 = b$.

$$\Rightarrow \alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$\alpha_1 + 2\alpha_2 = 1$

$2\alpha_1 + 4\alpha_2 = 1$

$0 = -1$ impossible

$\Rightarrow$ No, ~~a_1, a_2 are~~ b is not a linear comb. of a_1, a_2.

Recall, $b = x_1 a_1 + x_2 a_2 \dots + x_n a_n$

*/ $\boxed{\text{Theorem}}$ Consisty Theorem

1 Then, A linear system $\boxed{Ax = b}$ is consisent if and only if b can be written as a linear combination of the column vector of a.
i.e, $b = x_1 \cancel{*} a_1 \text{---} + x_n a_n$

same 3.6
$b \in$ span ~~column~~ column space

Eg Q12) let $A_{3 \times 4}$ if $b = 1a_1 + 1a_2 + a_3 + a_4$
is the system consistent?
If yes, what can you conclude about the # of solns?

Answer, The system is consistent since (1, 1, 1, 1) is a solution.
$Ax = b$
Since the system is underdetermined so, its has many
inf many solns.

1.3 Continue

Recall, $Ax = b$ is consistent iff $b = x_1 a_1 + \dots + x_n a_n$

E.g True or false?

① If $A_{3\times3}$ and $a_3 = a_1 - a_2$, then $Ax = 0$ has a unique solution False.

homogeneous

$a_1 - a_2 + a_3 = 0$

$(1, -1, -1)$ soln $\Rightarrow$ False.

Soln:- $0 = a_1 - a_2 + a_3$

$\Rightarrow (1, -1, -1)$ is a solution of $Ax = 0$

we also know that $(0, 0, 0)$ is a soln.

Then, it has infinitely many solns.

$a_1 = a_2$ $\qquad b = (1,1,1)$ $(2,0,1)$

② let $A_{4\times3}$ with $a_1 = a_2$ If $b = a_1 + a_2 + a_3$, then the system $Ax = b$ has infinitely many solutions. True

$b = 2a_1 + a_3$

Soln:- $b = a_1 + a_2 + a_3 \rightarrow$ consistent $(1,1,1)$ is a soln

Also $b = a_1 + a_1 + a_3$

$b = 2a_1 + a_3$ $\qquad (2,0,1)$ is another soln.

$\Rightarrow Ax = b$ has infinitely many solns.

---

* Transpose of a matrix

Def:- The transpose of a matrix A is defined by

$$A^T = (a_{ij})^T = (a_{ji})$$

E.g (1) $A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$

(2) $C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}$, $C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

Ex:- $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 6 \end{bmatrix}_{2\times3}$, $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 2 & 6 \end{bmatrix}_{3\times2}$

* $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$

Def:- A is symmetric if $A^T = A$ (see eg above)
A is called skew-symmetric if $A^T = -A$ (see eg above)

RMK:- The zero matrix is the only matrix that is both symmetric and skew-symmetric.

---

* Rules for transpose:-

① $(A^T)^T = A$ $\qquad (A^T)^T = A$
② $(\alpha A^T) = \alpha(A^T)$, α: scalar. $\qquad (\alpha A^T) = \alpha A^T$
③ $(A \pm B)^T = A^T \pm B^T$ $\qquad A^T \pm B^T$
④ $(ABC)^T = C^T B^T A^T$.(بالعكس) $\qquad (ABC)^T = C^T B^T A^T$

Eg T or F then

① let A be $m\times n$ matrix. Show that A^TA and AA^T are both symmetric True.

$(A^TA)^T = A^T(A^T)^T = A^TA$ $\quad \therefore (A^TA)^T = A^TA$
symmetric. ✓

$(AA^T)^T = AA^T$ ✓

② If A and B nxn symm matrices, then A+B is sym. True

Soln:- $(A \pm B)^T = A^T \pm B^T$

$= A \pm B$ ($\because$ A+B sym)

$\Rightarrow (A \pm B)^T = A \pm B$ symmetric

Note: The $\overset{\text{difference}}{\vee}$ sum of two symmetric Matrices of the same size (mxn) is symmetric.

③ If A nxn sym. matrix then αA is also sym for all $\alpha \in \mathbb{R}$. True

Soln:- $(\alpha A)^T = \alpha A^T = \alpha A$ ($\because$ A is symm)

$(\alpha A)^T = \alpha A \Rightarrow \alpha A$ is symmetric.

④ If A is sym. and skew-sym. Then A must be zero matrix True. ✓

Soln:-

$A^T = A$

$A^T = -A$

$\Rightarrow A = -A$

$2A = \mathbf{0}$

$A = \frac{1}{2}\,\mathbf{0}$ → matrix

[illegible]

the only

$A^T = A$

$A^T = -A$

$-A = A$

$\underline{\underline{0}}$

$\boxed{5}$ if A and B are sym. matrices, then $H = AB - BA$ is skew sym. True defined. يعني

Soln:

$$H^T = (AB - BA)^T$$
$$= (AB)^T - (BA)^T$$
$$= B^T A^T - A^T B^T$$
$$= BA - AB \quad \text{since } A, B \text{ are sym.}$$
$$= -(AB - BA)$$
$$= -H$$

$\boxed{6}$ let $A_{3\times3}$ Show that if A is skew-sym, then its diagonal entries must be zero True
$\downarrow (a_{11}, a_{22}, a_{33})$.

Soln:- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ Given that A is skew
$A^T = -A$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -e & -h & -i \end{bmatrix}$$

$a = -a \quad e = -e \quad i = -i$
$\Rightarrow a = 0 \quad e = 0 \quad i = 0$
the entries on the main diagonal $= 0$

* This is true for any square matrix any $A_{n\times n}$ ✓

$\boxed{7}$ let $A_{3\times3}$ show that if A is sym then its diagonal entries must be zero False.

$\boxed{1.4}$ Matrix Algebra

Theorem:-

For any scalars α and β and for any matrices A, B, C, Then the following is valid.

① $A + B = B + A$
② $(A+B)+C = A+(B+C)$
③ $(AB)C = A(BC)$
④ $A(B+C) = AB + AC$
⑤ $(A+B)C = AC + BC$
⑥ $(\alpha B)A = \alpha(BA)$
⑦ $\alpha(AB) = A(\alpha B)$
⑧ $(\alpha+B)A = \alpha A + \beta A$
⑨ $\alpha(A+B) = \alpha A + \alpha B$
⑩ $A^n = A \cdot A \cdots A$ (n-times)

E.g $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find A^{2019}

Soln:- $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

$A^4 = A^3 \cdot A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

$$A^{2019} = \begin{bmatrix} 2^{2018} & 2^{2018} \\ 2^{2018} & 2^{2018} \end{bmatrix}$$

• Identity matrix

always squared.
α in RREF.

$$I = (\delta_{ij}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

→ Main diagonal is 1 and anything else is 0

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_1 = [1]$$

RMK $AI = IA = A$, for any A.

• Matrix Inversion

Def:- An $n \times n$ - matrix A is said to be invertable or nonsingular if there exists a matrix B such that $AB = BA = I_n$

- The matrix B is called the inverse of A denoted by A^{-1}. If A^{-1} DNE, then A has no inverse. A in this case, is called a singular matrix or A is not invertable.

Eg let $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$. Verify that A and B are inverses of each other.

→

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} + \frac{12}{10} & \frac{4}{5} - \frac{4}{5} \\ -\frac{3}{10} + \frac{3}{10} & \frac{6}{5} - \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, B, A are inverses of each other.

---

* How to find $A^{-1}_{n\times n}$?

Ans For 2x2 matrix A :-

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\alpha = ad - bc \neq 0$

Then A^{-1} exists and $A^{-1} = \frac{1}{\alpha} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $\alpha = 0$ A^{-1} DNE (A is singular)

Pf: $AA^{-1} = \frac{1}{\alpha} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\leftarrow \quad = \frac{1}{\alpha} \begin{bmatrix} ad - bc & -ab + ba \\ cd - cd & -cb + da \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 \\ -1 & 2 \end{bmatrix}$

Similarly, $A^{-1}A = I_2$

E.g

Is $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ singular? $8-6 = 2 \neq 0$ det $\neq 0$ ∀ new

Soln:- $\alpha = 2 \neq 0$ $\Rightarrow$ A is nonsingular (invertable).

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ $\alpha = (3)(6) - (2)(9) = 0$

$\Rightarrow$ A is singular (A^{-1} DNE)

Theorem:- If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

PF:- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1}$ (since B^{-1} exists.

$= AA^{-1}$

$= I$ (since A^{-1} exists).

Similarly $(B^{-1}A^{-1})(AB) = I$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

Rmks:- ① In general, If $A_1, A_2, \ldots A_n$ are non sing $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ is nonsingular

$(A_1A_2 \ldots A_n)^{-1} = (A_n^{-1} \ldots A_1^{-1})$

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Eg $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

② If A and B are non singular then (A + B) are not necessarily non sing or sing.

Ex:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ non sing

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ non sing.

$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ sing $\boxed{\alpha = 0}$

③ Sing + Sing ⇒ Sing (False).

$A = \overset{\text{sing}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$ $B = \overset{\text{sing}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$

But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ non-sing.

H.W True or False?

① $A^2 - B^2 = (A-B)(A+B)$ (False).

$AB \neq BA$

$A^2 + \overbrace{AB + BA} + B^2$

② $(A+B)^2 = A^2 + 2AB + B^2$ (False).

③ If $AB = 0$ Then $A = 0$ or $B = 0$ (False).

let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

④ If $A^2 = 0$ Then $A = 0$ $\left(A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right)$ (False)

⑤ If $AB = AC$, Then $B = C$ (False)

⑥ If $A^2 = A$ Then $A = 0$ or $A = I$ (False).

⑦ If $AB = AC$ and A is non-sing then $B = C$ (True).

Both sides by A^{-1}

~~$A^{-1}B A^{-1}$~~

$(A^{-1}A)B = (A^{-1}A)C$

$IB = IC$

$\therefore \boxed{B = C}$

$A^2 = A$

$A^2 - A = 0$

$|A|^2 = |A|$

$|A|^2 - |A| = 0$

$|A|(|A| - 1) = 0$

$|A| = 0$

$|A| = 1$

} if $A^2 = A$ then $|A| = 0$ or $|A| = 1$

* Algebraic Rules for Inverse:-

① $(A^{-1})^{-1} = A$

any scalar

② $(\beta A)^{-1} = \frac{1}{\beta} A'$ $\quad \textcircled{\beta} \neq 0$ scalar.

③ If A is invertable then A^T is invertable and $(A^T)^{-1} = (A^{-1})^T$

④ $[(AB)^T]^{-1} = (A^{-1})^T (B^{-1})^T$

⑤ $(AB)^{-1} = B^{-1} A^{-1}$

Pf:- ② $(\beta A)\left(\frac{1}{\beta} A^{-1}\right) = \cancel{\beta}\frac{1}{\cancel{\beta}}(AA^{-1})$

$= I$

Similarly:- $\left(\frac{1}{\beta} A^{-1}\right) \beta A = I$

④ $[(AB)^T]^{-1} = [B^T A^T]^{-1} = (A^T)^{-1} (B^T)^{-1}$

$= (A^{-1})^T (B^{-1})^T$

$\downarrow$

By rule ③

Eg If $A_{n\times n}$ such that $A^2 = A$ (idempotent), then $I + A$ is non singular.

and $(I+A)^{-1} = I - \frac{1}{2}A$

Soln:- $(I+A)(I - \frac{1}{2}A)$

$= I + -\frac{1}{2}IA + AI - \frac{1}{2}A^2$

$= I - \frac{1}{2}A + A - \frac{1}{2}A$

$= I + 0 \quad = I$

Similarly, $(I - \frac{1}{2}A)(I + A) = I$ (prove)

H.W

Eg Q19 $A_{n\times n}$. Show that If $A^2 = 0$

Then $(I - A)$ is nonsing.

and $(I-A)^{-1} = I + A$

Eg Q14 A, B nxn matrices show that if $AB = A$ and $B \neq I$, the A must be singuler.

By contradiction

Suppose that A must be non sing.

$\boxed{1.5}$ Elementary Matrices

Def:- A matrix E is an elementary matrix if it is obtained from the identity by performing exactly one row operations.

- There are 3 types:-

$\boxed{1}$ Type I $E^{(1)}$ obtained by interching two rows.

E.g $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

Eg $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(1)}$ since we can $R_2 \leftrightarrow R_1 \Rightarrow I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Eg $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\boxed{2}$ Type II : $E^{(2)}$ obtained by multiplying a row of I by a non zero matrix.

Eg $\begin{bmatrix} 2019 & 0 \\ 0 & 1 \end{bmatrix}$ (Since $\frac{1}{2019} R_1 \rightsquigarrow I$)

$\begin{bmatrix} 0 & 1 \\ 2019 & 0 \end{bmatrix} \rightarrow$ not elementary at all.

$\boxed{3}$ Type III : $E^{(3)}$ obtained from I by adding a multiple of one row to another row.

Eg $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$-3R_3 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{(s)} = I$

~~Eg~~

Ex:- $E^{(1)} = \begin{bmatrix} a & \phi & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix} \updownarrow$

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \end{matrix}$$

$$= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \end{matrix}$$

$$AE = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{matrix} C_2 & C_1 & \end{matrix} \\ \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix} \end{array}$$

Theorem:- If E is an elementary matrix. Then, E is non singular and E^{-1} is an elementary matrix of the same type.

Eg If $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is $E^{(2)}$

$E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ is also $E^{(2)}$

Def:- ~~A~~ matrix B is a row equivalent to A if there exists a finite sequence $E_1, E_2, \ldots, E_n$ of elementary matices such that

$$B = (E_k E_{k-1} \ldots E) A.$$

→ one is enough.

In other words, B is row equivalent to A if B can be obtained from A by a finite nomber of row operations.

Example:- Show that $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is row equivalent to $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

A becomes B by ⇒ $R_2 - R_1$ on R_1. Now do this operation on I ⇒ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⇒ $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

∴ We need to find an elementary matrix E such that

$EA = B$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} R_2 - R_1$

$EA = B$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{E \cdot A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = B$$

Ex:- $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$

a) Find an elementary matrix ~~A~~ E such that $EA = B$
(In other words, B is row ∼ A).

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 + R_1$ $E \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

b) Find an elem. matrix F such that $FB = C$
(∴ C row ∼ to B)

Ans:- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -R_3 + R_2$ $F \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

ⓒ Is C row eq to A?

Ans, Yes

(a) $B = EA$ (b) $C = FB$

$\Rightarrow C = F(EA)$

$C = (FE)A \Rightarrow C$ is row $\sim A$

RMK:- ① If A is row eq to B, then B is row eq to A.

② If $A \overset{row}{\sim} B$ and $B \overset{row}{\sim} C$ then $A \overset{row}{\sim} C$

Proof:- ① $A \overset{row}{\sim} B$

~~$B =$~~ $A = (E_k \ldots E_1)B$, where $E_1 \ldots E_k$ are elementary matrices.

$B = (E_k \ldots E_1)^{-1} A$

$B = \underbrace{(E_1^{-1} \ldots E_k^{-1})}_{\downarrow} A$

we also know they're elementary.

$B \overset{row}{\sim} A$

② $A \overset{row}{\sim} B \Rightarrow A = (E_k \ldots E_1) B$ — ①
where E_i are elementary

Also, $B \overset{row}{\sim} C \Rightarrow B = (F_k \ldots F_1) C$ — ②
where F_i are elementary

substitute ② into ①

$$A = \left(\underbrace{(E_k \ldots E_1)(F_k \ldots F_1)}_{\text{elementary.}}\right) C$$

The

1.5 Continue. (Elementary Matrices)

Theorem:- very Important.

let A be an $n \times n$ matrix. The the following are equivalent:-

(a) A is nonsingular

(b) $Ax = 0$ has only the trivial solution ($x = 0$)

(c) A is row equivalent to I

Proof:- $a \Rightarrow b$

$b \Rightarrow c$

$c \Rightarrow a$

$$\begin{tikzcd} & a \arrow[dr] & \\ c \arrow[ur] & & b \arrow[ll] \end{tikzcd}$$

Later...

Application:-

Q31) Ex If A is 4×4 matrix and $a_1 + a_2 = a_3 + 2a_4$ Then A must be singular. (True)

$$0 = -a_1 - a_2 + a_3 + 2a_4$$

$\Rightarrow (-1, -1, 1, 2)$ is one solution of $Ax = 0$

$\Rightarrow Ax = 0$ has a non-trivial solution

$\hookrightarrow$ infinitely many solutions

$\xRightarrow{\text{Thm}}$ A is singular.

$\Rightarrow$ (Q15) let $A_{3\times3}$ and suppose that $2a_1 + a_2 - 4a_3 = 0$
How many solutions will the system $Ax = 0$ have? Is A non-singular? Explain.

Soln infinitely many solns & singular.

Proof:-

$(a \Rightarrow b)$

- Suppose $A_{n\times n}$ is non-singular, we need to prove $Ax = 0$ has only the trivial solution.

Indeed, A non singular $\Rightarrow A^{-1}$ exists.

$$\Rightarrow \quad A^{-1}(Ax) = A^{-1}(0)$$
$$IX = 0$$
$$\boxed{X = 0} \quad X \text{ is the soln of the System.}$$
✓

$(b \Rightarrow c)$

- Suppose that $Ax = 0$ has only the trivial solution we need to prove that $A \overset{\text{row}}{\sim} I$

Indeed, suppose not. That is $A \overset{\text{row}}{\not\sim} I$
So that RREF of A has a free variable.
$\Rightarrow Ax = 0$ has infinitely many solutions,
and thus contradicts the assumption.
So $A \overset{\text{row}}{\sim} I$

$(c \Rightarrow a)$

- Suppose that $A \overset{\text{row}}{\sim} I$ we need to prove that A is non-singular.

Indeed, $A \overset{\text{row}}{\sim} I \Rightarrow A = (E_k E_{k-1} \cdots E_1) I$
$\Rightarrow A = E_k E_{k-1} \cdots E_1$.
And we know E_i are nonsing (thm.).

$\Rightarrow$ A is a product of nonsingular matrices and Hence it must be non-singular.

Q.E.D

Corally (Special Case of the latter)

The system $Ax = b$, where $A_{n\times n}$, has a unique solution $\underset{\Longleftrightarrow}{\text{iff}}$ A is non-singular.

If A is singular $\rightarrow$ infinitely many
$\hookrightarrow$ no solution.

*Application:- How to find $A^{-1}_{n\times n}$? (if any)
$\downarrow$ a row = 0 0 0 no inverse.

$$[A : I] \xrightarrow[\text{RREF}]{\text{row operation}} [I : A^{-1}]$$

RMK:- If in the process of performing row operations on $[A:I]$ on row of A reduced to a zero row, then A^{-1} DNE.

E.g. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ Find A^{-1} (if any).

Soln $[A:I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \quad -2R_1 + R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 1 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 2 & 1 \end{array}\right] \begin{array}{l} -R_2+R_1 \\ \\ 2R_2+R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & -2 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] \begin{array}{l} \\ -2R_3+R_2 \\ \\ \end{array}$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

---

* Another way to solve the system:- using the inverse (if any)

⚠ Conditions :- ① Square ② the inverse exists

- Solving $Ax=b$, $A_{n\times n}$ using the inverse of A (if exists)

$$Ax=b \quad (A^{-1}A)x = A^{-1}b) \Rightarrow \boxed{x = A^{-1}b} \ \underline{\text{unique}}$$

E.g Solve
$$\begin{cases} x_1 + x_2 + 2x_3 = -2 \\ \qquad x_2 + 2x_3 = 3 \\ 2x_1 \qquad + x_3 = 0 \end{cases}$$

Sol
$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}}_{b}$$

A^{-1} from the previous example.

Now $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ $\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2-3+0 \\ -8-9+0 \\ 4+6+0 \end{bmatrix} = \begin{bmatrix} -5 \\ -17 \\ 10 \end{bmatrix}$$

$\Rightarrow x_1 = -5 \quad x_2 = -17 \quad x_3 = 10$ unique soln.

- Diagonal and Triangular matrices

Def:- Let $A_{n \times n}$ be a matrix

① If $a_{ij} = 0$, for $i > j$, then A is called upper triangular.

② If $a_{ij} = 0$, for $i < j$, then A is called lower triangular

③ A is said to be triangular if it is either lower or upper triangular

④ If $a_{ij} = 0 \ \forall i \neq j$, then A is called diagonal.

Examples:-

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$ upper triangular

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}$$ lower triangular.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ diagonal

↳ Both lower & upper

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ zero matrix :- Both diagonal & triangular

---

* LU Factorization

How to write $A = LU$ where L : unit lower triangular

main diagonal = 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 7 & 1 \end{bmatrix}$$

i.e. $L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}$

U :- Upper triagular (مش شرط unit)

$\boxed{1.5}$ continue.

$A = LU$, L: unit lower triangular necessarily
U: Upper triangular (not^{7}unit)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ X & 1 & 0 \\ X & X & 1 \end{bmatrix}$$

LU Factorization or LU decomposition.

Ex:- Compute the LU decomposition / factorization of the matrix

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$$

Soln: Step 1:- $U = ??$ REF * افترض

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix} \begin{matrix} \\ 2R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ 2R_2 + R_3 \end{matrix}$$

$\Rightarrow$ Upper Triangular.
Three operations

Step 2:- $L = \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ * each operation on Identity.

$I = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$

$\downarrow$

$E_1 = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ $2R_1 + R_2$ $\qquad E_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$ $-3R_1 + R_3$

$E_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}$ $2R_2 + R_3$ $\qquad L = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 3 & -2 & 1\end{bmatrix}$ Direct

$\boxed{E_3 E_2 E_1 A = U}$ Theorem.

$A = (E_3 E_2 E_1)^{-1} U$

$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$

Now, $\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right]$ $-2R_1 + R_2$

$E_1^{-1} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$ $\quad E_3^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}$

$L = E_1^{-1} E_2^{-1} E_3^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

LU decomposition doesn't always exist.

Ex ② Find the LU factorization

$$A = \begin{bmatrix} 2 & 4 & 2 \\ ① & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \qquad U = ??$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix} \quad -\frac{1}{2} R_1 + R_2$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \quad -2R_1 + R_3$$

نوصلها لصفر ب [illegible]

$$\rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow U. \qquad 3R_2 + R_3$$

check $\boxed{LU = A}$.

Remark (T or F)

① If A has an LU factorization. Then A is non singular iff $\boxed{L}$ is nonsing.
false → (U)

L is always non-sing. singularity بسبب U

② If A has LU-fact then A is nonsing. iff U is nonsing (True)

③ If A has LU-fact then A is row equavalent to U. (True).

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

④ L is always nonsing but U is not neccessarily (T)

⑤ A always has LU factorization (False), ← Quiz 1.3+1.4+1.5

* The transpose of the elementary matrix is an elementary matrix of the same type.
~ prove each type alone.

Chapter $\boxed{2}$: Determinants

2.1 The determinants of a matrix

Case $\boxed{1}$ $A = (a_{11})_{1\times1}$, then $\det(A) = |A| = a_{11}$

Ex:- $A = (5) \Rightarrow |A| = 5$

$A = (-5) \Rightarrow |A| = -5$

Case $\boxed{2}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - cb$

Ex:- $A = \begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}$ $|A| = (-2)(5) - (1)(4)$
$= -10 - 4 = \boxed{-14}$

VIP Theorem:- $A_{n\times n}$ is non-singular iff $|A| \neq 0$
$\Leftrightarrow$

A is singular iff $|A| = 0$
$\Leftrightarrow$

Ex:- $A = \begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}$ is non singular $\Rightarrow |A| = -14 \neq 0$

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $|A| = 0$ $\Rightarrow A$ is singuler.

Q18) $A = \begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix}$ is singular.

Find λ.

$|A| = 0 \quad (2-\lambda)(3-\lambda) - 12 = 0$

$(2-\lambda)(3-\lambda) = 12$

$6 - 2\lambda - 3\lambda + \lambda^2 - 12 = 0$

$(\lambda - 6)(\lambda + 1) = 0$

$\boxed{\lambda = 6} \quad \boxed{\lambda = -1}$

---

* Cofactors & Minors

Def:- let $A = (a_{ij})$ be $n \times n$ matrix and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. Then

Minor of $a_{ij} = |M_{ij}| = m_{ij}$

Cofactor of $a_{ij} = (-1)^{i+j} |M_{ij}| = A_{ij}$

Ex:- $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ Find m_{13}, A_{32}, A_{21}

Soln

$m_{13} = \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 12 - 5 = \boxed{7}$

$A_{32} = -\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 8$

$A_{21} = -\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} = -(30 - 16)$
$= -14$

2.1 continue (Determinants)

Recall, $A = (a_{ij})$, minor of $a_{ij} = m_{ij}$
Cofactor of $a_{ij} = A_{ij}$

Ex: $A = \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & 6 \\ -2 & 4 & -3 \end{bmatrix}$ find m_{12}, A_{32}, A_{22}

Soln: $m_{12} = \begin{vmatrix} -1 & 6 \\ -2 & -3 \end{vmatrix} = (-1)(-3) - (6)(-2)$
$= 3 + 12$
$= \boxed{15}$

$A_{32} = (-1)^{3+2} m_{32}$
$= - \begin{vmatrix} 3 & 5 \\ -1 & 6 \end{vmatrix} = -(18+5)$
$= \boxed{-23}$

$A_{22} = \begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix} = -9 + 10 = \boxed{1}$

Df:- Let $A_{n \times n}$, then $\det(A) = |A|$

$$= \begin{cases} a_{11} & \text{if } n = 1 \\ \boxed{\text{if } n > 1} \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} \end{cases}$$

Eg $A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = a_{11}A_{11} + a_{12}A_{12}$$
$$= a_{11}(-1)^2 M_{11} + a_{12}(-1)^3 M_{12}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

Eg $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$ Find $|A|$

$$= 3\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

= ~~3(~~

$$= -39 + 8 + 28$$
$$= -3 \quad \det \neq 0 \quad \text{non-singular.}$$

find $\left|\begin{array}{cccc} + & - & + & - \\ 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 7 & 8 & 2 & 0 \\ -2 & 1 & 5 & \boxed{6} \end{array}\right|$

$$= +1 \begin{vmatrix} 6 & 0 & 0 \\ 8 & 2 & 0 \\ 1 & 5 & 6 \end{vmatrix} + 0$$

$$= (+1)(+6)\begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix}$$

$$= 1 \times 6 \times (12)$$

$$= 72$$

|Any triangular| → product of triangular.

lower ↓ ↓

upper diagonal.

* Properties of determinants

[1] If Anxn matrix, then $|A^T| = |A|$

$|A^{-1}| \neq 0$

Eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = \boxed{-2} = |A^T|$

[2] If A is nxn triangular matrix, then

$|A| =$ the product of the diagonal entries.

$|I| = 1$

Eg $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 2 \\ 0 & 0 & 5 & 6 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} = (1)(6)(5)(1)(6)$

$= \underline{\underline{180}}$

[3] If Anxn matrix has a row or column consisting entirely of zeros, then $|A| = 0$

Ex:- $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 5 \end{vmatrix} = 0$

$\boxed{4}$ If $A_{n \times n}$ has two identical rows of colomns then $|A| = 0$

Eg
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 12 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

$\boxed{2.2}$ Properties of determinants

$\boxed{5}$ If a row or column of A is a multiple of another, then $|A| = 0$

Eg
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} \quad |A| = 0 \quad R_3 = 2R_1$$

$\boxed{6}$ $|I_n| = 1$ where I is nxn identity matrix

$\boxed{7}$ $|A+B| \neq |A| + |B|$ In general.

Eg $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$

$|A| = 0$ $|B| = 0$

$$|A+B| = \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \quad |A+B| = \boxed{2}$$

⑧ $|AB| = |A||B|$ (x) ⚠ they should be square of the same size (✓)

for $A_{n\times n}$ & $B_{n\times n}$

RMK: $A = LU$

$|A| = |L||U|$

$\underset{1\ (\text{unit})}{|L|}$ $\quad |A| = |U|$

singular → singular

non → non.

---

⑨ $|A^n| = |A|^n \quad n = 0, 1, 2, \dots \qquad A^0 = \text{Identity}.$

$$A = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} \text{ find } |A^{2019}|$$

$$|A^{2019}| = |A|^{2019} = (1)^{2019} = \boxed{1}$$

⑩ $|kA| = k^n |A|$, $A_{n\times n}$

Eg A, B, C are 3x3 matrices

$|A| = 9 \quad |B| = 2 \quad |C| = 3 \quad$ find $|4\underbrace{C^T B A^{-1}}_{3\times 3}|$

$$= 4^3 |C^T B A^{-1}|$$

$$= 4^3 |C^T||B||A^{-1}|$$

$$= 64\, |C||B| \frac{1}{|A|} = 64(3)(2)\frac{1}{9}$$

⑪ $|A^{-1}| = \frac{1}{|A|}$ provided A is non-singular $|A| \neq 0$

⑫ ~~If E is elementary~~

Ex: If $A^2 = A$ then $|A| = 0$ or $|A| = 1$

$A^2 = A$

$|A^2| = |A|$

$|A|^2 = |A|$

$|A|^2 - |A| = 0$

$|A|(|A| - 1) = 0$

$|A| = 0$
or $|A| = 1$

Ex:- If $A^TA = I$, then $|A| = \pm 1$

Theorem If $A^TA = I$ then A is non singular.

$|A^TA| = |I|$

$|A^T||A| = |I|$

$|A||A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$ q.e.d

Q16) If $A_{n\times n}$ is skew symmetric & n is odd. Show that A must be singular.

Proof:- A is skew-symmetric

$$A^T = -A$$

$$|A^T| = |-A|$$

$$|A| = (-1)^n |A| \qquad n \text{ is odd } (-1)^n = -1$$

$$|A| = -|A|$$

$$2|A| = 0$$

$$|A| = 0 \Rightarrow A \text{ is singular.}$$

symmetric $\Rightarrow$ $|A| = |A|$

could be $0 = 10$ or $2 = 2$] both singular or non singular.

same as (n) is even ↗

---

⑫ For elementary matrices:-

If E is elementary matrix, $|EA| = |E||A|$ ⚠

$$|E| = \begin{cases} -1 & \text{if } E^{(1)} \rightarrow \text{switch} \\ \alpha & \text{if } E^{(2)} \rightarrow \text{multiple} \\ 1 & \text{if } E^{(3)} \rightarrow \text{add a multiple} \end{cases}$$

بشكل عام مثلاً $2R_1+R_2$ ما بأثر
بالـ det. ليس فقط ←

E.g $E^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |E^{(1)}| = -1$

$$E^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2019 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |E^{(2)}| = 2019$$

$$E^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |E^{(3)}| = 1$$

Remark:- We can use Gaussian Elimination to find $|A|$ by transforming A to triangular matrix which is easier to compute.

Ex:- Find

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$$

لو اضطريت ابدل . بضرب
سالب واحد . وهكذا
لانه نفس العلامة بيصير
عالثاني ،

$$= \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix} \quad R_1 + R_4$$

Now triangular. $= 1 \times 3 \times 2 \times 5$

$= \boxed{30}$

Ex:-

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}$$

ما بقدر اقسم جوا
leading one.
← مش augment!

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \begin{matrix} \\ -2R_1 + R_3 \\ -3R_1 + R_3 \end{matrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix}$$

كل تبديلة
بسالب

$$= - \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} = -(2)(-6)(-5)$$

$= \boxed{-60}$

Ex:- If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

Find $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$

لا بدي اطلع 2 بس
من صف بتطلع زي ما هي
عنا صيغة $(\alpha)^n$.
??

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad R_3 - R_1$$

$$= 2 \times 5 = \boxed{10}$$

---

if A is nonsing $\Rightarrow |A^{-1}| = \frac{1}{|A|}$

If A is non sing $AA^{-1} = I$

$$\Rightarrow |AA^{-1}| = |I|$$

$$|A|\,|A^{-1}| = I$$

$$|A^{-1}| = \frac{1}{|A|} \quad ✓$$

q.e.d.

Theorem:- If A and B are $n \times n$ matrices, then $|AB| = |A||B|$

Proof:-
- If B is singular
  $\Rightarrow AB$ is also singular
  (see Ex. 18 sec 1.5)

$\Rightarrow |AB| = 0$

Now, $|A||B| = |A| 0 = 0$

$\Rightarrow |A||B| = |AB|$

- If B is non singular
  $\Rightarrow B = E_k E_{k-1} \ldots E_1 I$
  where $E_k\ E_{k-1} \ldots E_1$: are elementary matrices.

$$|AB| = |A\ E_k\ E_{k-1} \ldots E_1|$$
$$= |A||E_k| \ldots |E_1|$$ بتوزع لانه اساسيا
$$= |A|\ |E_k \ldots E_1|$$
$$= |A|\ |B|$$

q.e.d

2.3 Additional topic & Applications

the adjoint of a matrix

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & & A_{nn} \end{bmatrix}^T$$

whee $A_{ij} = (-1)^{i+j} M_{ij}$

Ex:- If $A = \begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix}$ find adj (A)

Soln:- $\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$

$A_{11} = +|6| = 6$

$A_{12} = -|4| = -4$

$A_{21} = -|3| = -3$

$A_{22} = +|-1| = -1$

$$\operatorname{adj}(A) = \begin{bmatrix} 6 & -4 \\ -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ -4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times \underbrace{\begin{bmatrix} 6 & -3 \\ -4 & -1 \end{bmatrix}}_{\text{adjoint}}$$

Scanned by CamScanner

Rmk:- ① $A \cdot (\text{adj}(A)) = |A| I$ *

② If the matrix A is non singular

$$\frac{1}{|A|} A (\text{adj} A) = I$$

$$A\Big(\underbrace{\frac{1}{|A|} \text{adj}(A)}_{\text{this is the inverse.}} \Big) = I$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{|A|} \text{adj} A}$$

* Another way
* to find the inverse

Discussion (1.5 + 2.1 + 2.2)

$(A+B)^2 = A^2 + 2AB + B^2$ (false)

$(A+B)(A+B)$

$A^2 + \underbrace{AB + BA}_{} + B^2$

$AB + BA$ in general. $\underset{m\times n}{A}\underset{n\times 1}{x} = \underset{m\times 1}{b}$

or equals if

$AB = BA$

$\boxed{1.5}$ (17, 18, 22, 29, 30, 31, 32)

17) $A, B_{n\times n}$ $C = A - B$

Show that if $AX_0 = BX_0$, $X_0 \neq \vec{0}$

Then C must be singular.

det بنفعش اوزع لازم يكونوا [illegible]

Pf:- We need to prove $CX = 0$ has nontrivial solution.

Now $AX_0 = BX_0$ $AX_0 - BX_0 = 0$

$(A-B)X_0 = 0$ $X_0 \neq 0$

$CX_0 = 0$ $X_0 \neq 0$

$\Rightarrow CX = 0$ has nontrivial Solution.

$\Rightarrow$ C is singular (theorem).

---

18) $A, B_{n\times n}$ let $C = AB$ prove that if B is singular then C must be singular. (Theorem)

Pf:- Since B is singular $\rightarrow \exists\, X_0 \neq 0$ such that $BX_0 = 0$

column vector

$A(BX_0) = A0$

$(AB)X_0 = 0$, $X_0 \neq 0$

$CX_0 = 0$, $X_0 \neq 0$

$\Rightarrow CX = 0$ has nontrivial solution

$\Rightarrow C$ is singular.

$C = AB$

$|C| = |A||B|$

$|B| = 0$

$|A||B| = 0 \rightarrow$ singular.

Q) 11c $A, B_{n\times n}$, $|AB| = |BA|$ $\boxed{2.1}$

Pf: $|AB| = |A||B|$

$= |B||A|$ real numbers.

$= |BA|$

$\boxed{22}$ Show that if A is a symmetric & nonsing then A^{-1} is symmetric.

Pf: Given A^{-1} exists.

& $A^T = A$

we need to prove $(A^{-1})^T = A^{-1}$

Indeed, $(A^{-1})^T = (A^{-1})^T I$

$= (A^{-1})^T A^{-1} A$

$= (A^{-1})^T A A^{-1}$ sine A^{-1} exist.

$= (A^{-1})^T A^T A^{-1}$ since A symm.

$= (AA^{-1})^T A^{-1}$

$= (I)^T A^{-1}$ $I^T = I$

$= I A^{-1}$

$= A^{-1}$

---

or $AA^{-1} = I$

$(AA^{-1})^T = I^T$

$(A^{-1})^T A^T = I$

so $(A^T)^{-1} = (A^{-1})^T$.

switch يعني بدل

$\boxed{29}$ If $A \overset{row}{\sim} I$ and $AB = AC$ then $B = C$

Sdn:- since $A \overset{row}{\sim} I \Rightarrow A$ is nonsingular (A^{-1} exists) (Theorem)

$\Rightarrow A^{-1}AB = A^{-1}AC$

$(A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow B = IC$

$\boxed{B = C}$

$\boxed{30}$ E & F are elementary matrices then EF is nonsingular.

Proof:- Since E and F are nonsingular then EF is nonsingular (Theorem)

$\boxed{31}$ $A_{4\times4}$, $a_1 + a_2 = a_3 + 2a_4$, then A is singular

Sol $a_1 + a_2 - a_3 - 2a_4 = 0$

$\Rightarrow (1, 1, -1, -2)^T$ is one Sdn of $Ax = 0$

$\Rightarrow Ax = 0$ has infinitely many solns $\Rightarrow$ Singular (theorem)
non-trivial sdn.

$\boxed{32}$ $A \overset{row}{\sim} B$ $\cancel{B \overset{row}{\sim} C}$
& $A \overset{row}{\sim} C$ then $A \overset{row}{\sim} B + C$ (False)

sdn:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$A \overset{row}{\sim} B$
$A \overset{row}{\sim} C$ but $B + C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\underset{\substack{\downarrow \\ \text{non singular}}}{A} \overset{row}{\not\sim} \underset{\substack{\downarrow \\ \text{singular}}}{B+C}$

$\boxed{2.2}$ 6 ✓

$\boxed{5}$ $A_{n \times n}$ α scalar

$\det(\alpha A) = (\alpha)^n |A|$ prove

Pf: $\det(\alpha A)$

$$\begin{aligned} |\alpha A| &= |\alpha I A| \\ &= |(\alpha I) A| \\ &= |\alpha I| |A| \\ &= \underbrace{\alpha \ldots \alpha \ldots \alpha}_{n\text{-times}} |A| \\ &= \alpha^n |A| \end{aligned}$$

$\boxed{6}$ $|A^{-1}| = \frac{1}{|A|}$ if A is nonsing (done)

$AA^{-1} = I$

See your notes. $|A||A^{-1}| = 1$ $|A^{-1}| = \frac{1}{|A|}$

$\boxed{14}$ $A, B_{n \times n}$ prove AB is nonsingular iff A, B are both nonsingular.

Soln $(\Leftarrow)$ if AB nonsing $\rightarrow$ A & B are nonsingular.

$|AB| \neq 0$

$|A||B| \neq 0$ $\Leftrightarrow$ A is nonsing and B is nonsing.

$\Rightarrow |A| \neq 0$ and $|B| \neq 0$

$\boxed{2.3}$ Continue.

Recall,

$$\operatorname{adj}A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}^T$$

$A_{ij} = (-1)^{i+j}$ Minor of a_{ij}

Rmk ① $A(\operatorname{adj}A) = |A| I$ always true (singular or not)

important ② if $|A| \neq 0$ (A nonsing)

$$\Rightarrow A^{-1} = \frac{1}{|A|}\operatorname{adj}A$$

Eg If $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Find 1) adj A 2) A^{-1}

Soln:- $A_{11} = +\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 6-4 = \boxed{2}$ $A_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = \boxed{1}$

$A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = \boxed{-7}$ $A_{22} = +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = \boxed{4}$

$A_{13} = +\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = \boxed{4}$ $A_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \boxed{-3}$

$A_{31} = +\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = \boxed{-2}$ $A_{33} = +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = \boxed{1}$

$A_{32} = -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = \boxed{2}$

$$\operatorname{adj} A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = 2(2) - 1(-7) + 2(4)$$
$$= 4 - 7 + 8 = \boxed{5}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{-7}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-3}{5} & \frac{1}{5} \end{bmatrix}$$

Application $\boxed{2}$:- Solving a system using determinants.

* Cramer's Rule

How to solve the system $Ax=b$ by Cramer's Rule?

Theorem let $A_{n \times n}$ be nonsingular & let $B \in \mathbb{R}^n$ & let A_i be the matrix obtained by replacing the i^{th} column of A by b.

If $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the unique solution → because A is nonsingular (Corollary).

of $Ax=b$ then $x_i = \frac{|A_i|}{|A|}$, $i = 1, \dots, n$

Ex:- Use Cramer's rule to solve

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 9$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$|A| = \boxed{-4}$ & $n \times n$ ✓

We can apply Cramer's Rule.

$$A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} \Rightarrow |A_1| = \boxed{-4}$$

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} \Rightarrow |A_2| = \boxed{-4}$$

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} \Rightarrow |A_3| = \boxed{-8}$$

$$x_1 = \frac{-4}{-4} = \boxed{1}$$

$$x_2 = \frac{|A_2|}{|A|} = \boxed{1}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-4} = \boxed{2}$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2.3 Outline Questions.

8 $A_{n \times n}$ nonsing $n > 1$

Show that $|\mathrm{adj} A| = |A|^{n-1}$ Important theorem

A nonsing $\Rightarrow$ $A^{-1} = \frac{1}{|A|} \mathrm{adj} A$

$$\mathrm{adj} A = |A| A^{-1}$$

$$|\mathrm{adj} A| = \Big| |A| A^{-1} \Big|$$

$$= |A|^n |A^{-1}|$$

$$= |A|^n \frac{1}{|A|}$$

$$|\mathrm{adj} A| = |A|^{n-1}$$

q.e.d

if $n = 1$ $A = (a_{11})$

$\mathrm{adj}(A) = (A_{11})^T \rightarrow$ no matrix

---

10 if A is nonsingular then adj A is also nonsingular.

$$(\mathrm{adj} A)^{-1} = |A^{-1}| A = \mathrm{adj} A^{-1}$$

or $\frac{A}{|A|}$

---

11 if A is singular then adj A is singular.

$$|\mathrm{adj} A| = \underset{\text{zero}}{|A|}^{\boxed{n-1} > 0}$$

$\boxed{12}$ Show that if $|A| = 1$, then $\text{adj}(\text{adj} A) = A$

Soln:- $\text{adj}(\text{adj} A)$

$= \text{adj}(|A| A^{-1})$ let $B = |A| A^{-1}$

$= \text{adj} B$

$= |B| B^{-1}$

$= \left| |A| A^{-1} \right| (|A| A^{-1})^{-1}$

$= |A|^n |A^{-1}| \frac{1}{|A|} (A^{-1})^{-1}$

$= |A|^n \frac{1}{|A|} \frac{1}{|A|} A$

$= |A|^{n-1} A \rightarrow$ for nonsingular A

$= 1 \cdot A = \boxed{A}$

theorem: if A is nonsingular then $\text{adj}(\text{adj} A) = |A|^{n-2} A$

Remark:- $|\text{adj}(\text{adj} A)| = \left| |A|^{n-2} A \right|$ for $2 \times 2 = |A|$

$= (|A|^{n-2})^n |A|$

$= |A|^{n^2-2n} |A|$

$= |A|^{n^2-2n+1}$

$= |A|^{(n-1)^2}$

3.1

Chapter $\boxed{3}$: Vector Space

Df. A vector space is a set of elements together with the operations of addition and scalar multiplication such that the following axioms are satisfied:-

C1: If $x \in V$ and α is scalar, then $\alpha x \in V$
(closed under scalar multiplication).

C2: If $x, y \in V$, then $x+y \in V$ (closed under addition)

A1: $x+y = y+x \quad \forall x, y \in V$

A2: $(x+y)+z = x+(y+z), \ \forall x, y, z \in V$

A3: $\exists$ an element $0 \in V$ such that $x+0 = x, \forall x \in V$ 0 is an element

A4: $\forall x \in V, \exists -x \in V$ such that $x+(-x) = 0$

A5: $\alpha(x+y) = \alpha x + \alpha y$, for each α scalar, $x, y \in V$

A6: $(\alpha+\beta)x = \alpha x + \beta x$, for each α, β scalar, $x \in V$

A7: $(\alpha\beta)x = \alpha(\beta x), \ \forall x \in V, \ \alpha, \beta$ scalar

A8: $1x = x, \ \forall x \in V$ any 1 not necessarily $\in V$

Notation $(V, +, \cdot)$

Example $\boxed{1}$ let $V = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$ (x-y plane)
with usual $+$ and $\cdot$ that is

$$(a,b)+(c,d) = (a+c, b+d)$$
$$\alpha(a,b) = (\alpha a, \alpha b)$$

Show that $(V, +, \cdot)$ is a vector space.

Pf. C① α scalar $(a,b) \in V$
$\alpha(a,b) = (\alpha a, \alpha b) \in V$

C② let $(a,b), (c,d) \in V$
now $(a,b)+(c,d) = (a+c, b+d) \in V$

A1: $(a,b)+(c,d)=(c,d)+(a,b)$

$$\begin{aligned} L.H.S &= (a,b)+(c,d) \\ &= (a+c,\ b+d) \\ &= (c+a,\ d+b) \\ &= (c,d)+(a,b) \end{aligned}$$

A2:

Example②:- let $V=\mathbb{R}^n$ with usual addition

$(x_1, x_2 \dots x_n)+(y_1, \dots y_n)=(x_1+y_1 - - - x_n+y_n)$

and scalar multiplication

$\alpha(x_1, \dots x_n)=(\alpha x_1, \dots \alpha x_n)$

is a vector space $\rightarrow (\mathbb{R}^n, +, \cdot)$

example①:- A2: $\left[(a,b)+(c,d)\right]+(e,f)=(a,b)+\left[(c,d)+(e,f)\right]$ ✓

A3: let $\begin{pmatrix}0\\0\end{pmatrix}=0 \in \mathbb{R}^2$

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix} \checkmark \quad \forall \begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2$$

A4: $\forall \begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2$

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}-a\\-b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

A5: $\alpha\left[\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right]\overset{?}{=}\alpha\begin{pmatrix}a\\b\end{pmatrix}+\alpha\begin{pmatrix}c\\d\end{pmatrix}$

$$L.H.S = \alpha\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} \alpha a+\alpha c \\ \alpha b+\alpha d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} + \begin{pmatrix} \alpha c \\ \alpha d \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a \\ b \end{pmatrix} + \alpha\begin{pmatrix} c \\ d \end{pmatrix} = R.H.S$$

<u>A6:-</u> $(\alpha+\beta)\begin{pmatrix} a \\ b \end{pmatrix} \overset{?}{=} \alpha\begin{pmatrix} a \\ b \end{pmatrix} + \beta\begin{pmatrix} a \\ b \end{pmatrix}$

$$L.H.S = \begin{pmatrix} (\alpha+\beta)a \\ (\alpha+\beta)b \end{pmatrix} = \begin{pmatrix} \alpha a+\beta a \\ \alpha b+\beta b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} + \begin{pmatrix} \beta a \\ \beta b \end{pmatrix} = \alpha\begin{pmatrix} a \\ b \end{pmatrix} + \beta\begin{pmatrix} a \\ b \end{pmatrix} = R.H.S$$

<u>A7:-</u> $(\alpha\beta)\begin{pmatrix} a \\ b \end{pmatrix} \overset{?}{=} \alpha\left(\beta\begin{pmatrix} a \\ b \end{pmatrix}\right)$

$$(\alpha\beta)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (\alpha\beta)a \\ (\alpha\beta)b \end{pmatrix} = \begin{pmatrix} \alpha(\beta a) \\ \alpha(\beta b) \end{pmatrix}$$

$$= \alpha\begin{pmatrix} \beta a \\ \beta b \end{pmatrix} = \alpha\left(\beta\begin{pmatrix} a \\ b \end{pmatrix}\right) = R.H.S$$

A8 $\Rightarrow 1x = x \quad 1\cdot\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ ✓

$\Longrightarrow (\mathbb{R}^2, +, \cdot)$ is a vector space.

Ex:- $(\mathbb{R}, +, \cdot)$

Ex ③:- $V = \{(x, 1) : x \in \mathbb{R}\}$ under + & · (usual)

is not a vector space.

C1 $\begin{pmatrix} 2019 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in V$

but $\begin{pmatrix} 2019 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2021 \\ 2 \end{pmatrix} \notin V$

since $2 \neq 1$

Ex $\boxed{4}$ $V = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R}\right\}$ +, · (usual)

is a vector space.

Ex⑤ let $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

Q10)

under $+$ & $\cdot$ as:-

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ 0 \end{pmatrix}$$

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$$

Solu:- No

A6 $(\alpha + \beta)x = \alpha x \oplus \beta x$

$\alpha = 2 \quad \beta = 5 \qquad x = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$$(\alpha + \beta)x = (2+5)\begin{pmatrix} 4 \\ 6 \end{pmatrix} = 7\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 28 \\ 42 \end{pmatrix}$$

$$\alpha x + \beta x = 2\begin{pmatrix} 4 \\ 6 \end{pmatrix} + 5\begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 12 \end{pmatrix} + \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 28 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 28 \\ 42 \end{pmatrix}$$

A6 Fails.

Ex ⑥ $V = \{ f(x) : \deg(f) = 3 \}$

under $+$, $\cdot$ usual

x^4 $\quad -x^4 + x^3$

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)x = \alpha f(x)$

Answ No.

CI $f(x) = x^3 \qquad g(x) = -x^3 + 4$

$f, g \in V$ but

$f+g = x^3 - x^3 + 4 = 4 \notin V$

since $\deg(f+g) \neq 3$

More examples. (Very Important)

① $M_{n \times m} = \mathbb{F}^{n \times m}$ is the set of all $m \times n$ matrices under usual $+$ and $\cdot$ is a vector space.

② The set of all $\overset{\text{range}}{\text{real}}$ valued functions under $+$ and $\cdot$ as follows:-

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha f(x)$

is a vector space

③ $\underset{\text{continuous}}{C}\,\underset{\text{on } a,b}{[a,b]} = \{ f : [a,b] \longrightarrow \mathbb{R} : f \text{ is continuous on } [a,b] \}$

كل الاقترانات المتصلة على فترة معينة

under + and.

cont + cont → cont.

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha f(x)$

is a vector space.

0 = zero function

$g(x) = 1$ identity.

↑ special case.

④ $C^n [a,b] = \{ f : [a,b] \rightarrow \mathbb{R}$ such that $\overset{\text{مشتقة}}{f^{(n)}}$ is cont on $[a,b] \}$

is a vector space under + &. as above.

⑤ $P_n = \{ f : f(x) = a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_0, a_1, \ldots, a_{n-1} \in \mathbb{R} \}$

$\deg(P_n) = n-1$ or less under + &. (usual) is a vector space.

Ex:- $P_3 = \{ f : f(x) = a x^2 + bx + c \}$ $a, b, c \in \mathbb{R}$

$P_1 = \{ f : f(x) = c$ constant$\}$

⑥ $\mathbb{Q} = \{ \frac{a}{b} : a, b$ integers , $b \neq 0 \}$

under usual + &.

is not a vector space.

Sol $\alpha = \sqrt{2}$ scalar $\frac{4}{5} \in \mathbb{Q}$

but $\sqrt{2}\,\frac{4}{5} = \frac{4\sqrt{2}}{5} \notin \mathbb{Q}$

$\overline{\mathbb{Q}}$ = irrational numbers is not a vector space since $\sqrt{2}, -\sqrt{2} \in \overline{\mathbb{Q}}$

but $\sqrt{2} + -\sqrt{2} = 0 \notin \overline{\mathbb{Q}}$

⑦ Integers $= \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is not a vector space.

since $\frac{1}{2}$ is a scalar $5 \in \mathbb{Z}$ but $\frac{1}{2} \cdot 5 = \frac{5}{2} \notin \mathbb{Z}$

Theorem: If V is a vector space and $\vec{x} \in V$, then

(i.) $0\vec{x} = \vec{0}$

(ii) $\vec{x} + \vec{y} = \vec{0} \Rightarrow \vec{y} = -\vec{x}$

(iii) $(-1)\vec{x} = -\vec{x}$. ? // $\big((-1)(1)\big)\vec{x} = 1\big(-1(\vec{x})\big)$

$= -\vec{x}$

Proof:- (i) A6 & A8

$$\vec{x} = 1\vec{x} = (1+0)\vec{x}$$
$$= 1\vec{x} + 0\vec{x} \quad (A6)$$

$$-\vec{x} + \vec{x} = \underbrace{-\vec{x} + 1\vec{x}} + 0\vec{x}$$

$$\vec{0} = \vec{0} + 0\vec{x}$$
$$\vec{0} = 0\vec{x}$$

(ii) & (iii) left to the reader.

$\boxed{3.2}$ Subspace & Spanning set

→ Subspace

Definition:- A nonempty subset S of a vector space V is called subspace iff

① $x+y \in S \ , \ \forall x,y \in S$

② $\alpha x \in S \ , \ \forall \alpha \in \mathbb{R}$ and $x \in S$

Theorem: let S be a subspace of a vector $\vec{0} \in S$

contrapositive: if $\vec{0} \notin S$ then S is not a subspace.

E.g $\{(x,1) : x \in \mathbb{R}\}$ not a subspace of $\mathbb{R}^{(2)}$

Note:- if the zero is there, we can't tell.

RMK:- let S be a subset of V

If $\vec{0} \notin S$, then S is not a subspace.

Ex:- Subspace or not?

① $S = \{(a,b)^T : a+b=1 \ , a,b \in \mathbb{R}\}$. Is S a subspace of $\mathbb{R}^2$?

comes from $\mathbb{R}^{(2)}$. The zero is not there $(0,0)^T \notin S$

⇒ S isn't a subspace — since $0+0 \neq 1$

② Also, $(1,0)\ (0,1) \in S$ but $(1,0)+(0,1)=(1,1) \notin S$

since $1+1 \neq 1$

② $S = \{(1,y)^T : y \in \mathbb{R}\}$ $V = \mathbb{R}^2$

No, $(1,5), (1,2) \in S$

but $(1,5)+(1,2) = (2,7) \notin S$

"$2 \neq 1$"

③ $S = \{(0,y,z) : y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

since (i) $(0,0,0) \in S \Rightarrow S \neq \phi$

(ii) let $(0,y,z), (0,a,b) \in S$

$(0,y,z)+(0,a,b) = (0, y+a, z+b) \in S$

(iii) let α be scalar & $(0,y,z) \in S$

$\alpha(0,y,z) = (\overset{\swarrow}{0}, \alpha y, \alpha z) \in S$

$\therefore$ (i)-(iii) $\Rightarrow$ S is subspace.

④ $S = \{A_{n\times n} : |A| = 0\}$, $V = \mathbb{R}^{n\times n}$

is not a subspace

since $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$

but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$ "$|A| = 1 \neq 0$"

⑤ $S = \{A_{n\times n} : |A| \neq 0\}$

$V = \mathbb{R}^{n\times n}$ is not a subspace. (H.W zero is not there.

$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in S$,

but $A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $|A+B| = 0 \Rightarrow A+B \notin S$

---

⑥ let S be the set of all symmetric $n\times n$ matrices
that is $S = \{A_{n\times n} : A^T = A\}$
subspace.

① $0 \in S$ since $0^T = 0$ $S \neq \phi$

② let $A, B \in S$ i,e, $A^T = A$ & $B^T = B$

$(A+B)^T = A^T + B^T$

$= A+B \Rightarrow A+B \in S$

③ let α be scalar ~~of~~ & $A \in S$ " $A^T = A$ "

$(\alpha A)^T = \alpha A^T = \alpha A$

$\Rightarrow \alpha A \in S$

$\therefore S$ is a subspace

---

⑦ the set of all skew-symmetric matrices is
a subspace

⑧ let $S = \{A_{2\times 2} : a_{12} = -a_{21}\}$

Is S subspace of $\mathbb{R}^{2\times 2}$

Soln $A \in S \Rightarrow A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$

Yes, (i) $S \neq \phi$

since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$

(ii) let $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ & $\begin{bmatrix} d & e \\ -e & f \end{bmatrix} \in S$

Now, $A + B = \begin{bmatrix} a+d & b+e \\ -b-e & c+f \end{bmatrix} \in S$

(ii) let α be scalar & $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \in S$

then,

$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \in S$

$\therefore$ S is a subspace.

$\boxed{9}$ the set of all polys in P_4 of even degree

$$V = \left\{ f(x) = ax^3 + bx^2 + cx + d \quad a,b,c,d \in \mathbb{R} \right\} = P_4$$

$$S = \{ f \in V : \deg(f) = \text{even} \}$$

No $f = 1 - x^2 \quad g = 1 - 2x + x^2 \in S$

but $f + g = 2 - 2x \notin S$ (odd degree).

$\boxed{10}$ the set of all polys in P_4 having at least one real root

Ans No, $f = 1 - x^3 \quad g = 1 + x^3 \in S$

but $f + g = 1 - x^3 + 1 + x^3$

$= 2 \neq 0$ Has no roots. $\notin S$.

Theorem:- let S & T be subspaces of a vector space V. Then ① $S \cap T$ is a subspace.

② $S \cup T$ is not a subspace.

?③ $S + T = \{ x : x = s + t,\ s \in S,\ t \in T \}$. is a subspace. Q11/14 Eg $(5,6) = (5,0) + (0,6)$

Pf:- ① (i) $S \cap T \neq \emptyset$

since $0 \in S$ & $0 \in T$

$\Rightarrow 0 \in S \cap T$

(ii) let $x, y \in S \cap T$

$\Rightarrow x, y \in S$ and $x, y \in T$

$\Rightarrow x + y \in S$ & $x + y \in T$ (S & T are subspaces)

$\Rightarrow x + y \in S \cap T$

(iii) let α scalar $x \in S \cap T$

$x \in S$ & $x \in T$

$\Rightarrow \alpha x \in S$ & $\alpha x \in T$ (Subspaces)

$\alpha x \in S \cap T$

q.e.d

$\boxed{2}$ let $V = \mathbb{R}^2$

$S = \{(0,y) : y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

$T = \{(x,0) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

$S \cup T = \{(x,y) : x \text{ or } y \text{ is } 0\}$

$\underset{\overset{\downarrow}{\in S}}{(0,1)}, (1,0) \in T$

$(0,1) + (1,0) = (1,1) \notin S \cup T$

* The Null space of a matrix

Def:- let A be $m \times n$ matrix. The null space of A is

$$N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$$ ← homo ال حل
null. set

Ex:- If $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2 \times 4}$

find $N(A)$

Sol: Solve $Ax = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} ① & 1 & 1 & 0 & 0 \\ 0 & ① & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix}$$

x_1 & x_2 → leading
x_3 & x_4 → free.

$x_3 = r$ $x_4 = t$ $x_1 = x_3 - x_4 = r - t$
$x_2 = -2x_3 + x_4 = -2r + t$

$$N(A) = \left\{ x \in \mathbb{R}^4 \ x = \begin{bmatrix} r-t \\ -2r+t \\ r \\ t \end{bmatrix}, r, t \in \mathbb{R} \right\}$$

Theorem:- let A be $m \times n$ matrix then $N(A)$ is a subspace of $\mathbb{R}^n$

$$N(A) = \{x \in \mathbb{R}^n ; Ax = 0\}$$

Pf:- (i) $\cancel{S}N(A) \neq \phi$

(i) $N(A) \neq 0$ since $A0 = 0 \Rightarrow 0 \in N(A)$

(ii) let $x, y \in N(A)$ $\quad Ax = 0$

$Ay = 0$

Now $A(x+y) = Ax + Ay$

$= 0 + 0 = 0$

$\Rightarrow x + y \in N(A)$

(iii) let α be scala & $x \in N(A) \Rightarrow Ax = 0$

$A(\alpha x) = \alpha(Ax) = 0$

$\alpha x \in N(A)$ $\quad$ q.e.d

---

* Linear Combinations.

Def: let V be a vector space & let $v_1, v_2, \dots v_n \in V$

$c_1, c_2, \dots c_n \in \mathbb{R}$

then $c_1 v_1 + \dots + c_n v_n = \cancel{V}$

is called a linear combination. $\quad (2,7)$ linear com of $(1,0), (0,1)$

* $\text{Span}(v_1, v_2, \dots, v_n)$ = the set of all linear combination of $v_1, v_2, \dots v_n$.

Example:- Is $v = \begin{pmatrix}2\\3\end{pmatrix}$ a liner combination of $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 \begin{pmatrix}1\\1\end{pmatrix}$?

Soln:- let $v = c_1v_1 + c_2v_2$

$$\begin{pmatrix}2\\3\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$$

$2 = c_1 + c_2$ $\boxed{c_1 = -1}$

$\boxed{3 = c_2}$

$$\begin{pmatrix}2\\3\end{pmatrix} = -1\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix}$$

$$\boxed{v = -v_1 + 3v_2}$$

Example:- Is $f(x) = x \in \text{Span}(1, 3x)$?

let $f(x) = x = c_1 \cdot 1 + c_2 \cdot 3x$

$x : 1 = 3c_2 \Rightarrow \boxed{c_2 = \frac{1}{3}}$

const $0 = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\underset{v}{x} = 0\underset{v_1}{(1)} + \left(\tfrac{1}{3}\right)\underset{v_2}{(3x)}$$

x is a linear combination of $\{1 , 3x\}$

that is $x \in \text{Span}(1, 3x)$

3.2 Continue

Ex:- Span (e_1, e_2) in $\mathbb{R}^3$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Eg $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\rightarrow$ ith

Span $(e_1, e_2) = ?$

Let $V \in$ span (e_1, e_2)

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \qquad \begin{aligned} c_1 &= x \\ c_2 &= y \end{aligned}$$

$$\text{Span}(e_1, e_2) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$ x-y plane

Ex $$\text{Span}(e_1, e_3) = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$$ x-z plane

$\text{Span}(e_1, e_2, e_3) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} = \mathbb{R}^3$

$(2,4,6) = 2(1,0,0) + 4(0,1,0) + 6(0,0,1)$

Def: let V be a vector space
A set $v_1, v_2, \dots, v_n \in V$ is called a spanning set iff $V = \text{span}(v_1, v_2, \dots, v_n)$

Eg $\text{span}(e_1, e_2, e_3)$. ← spanning set.

Ex: $\{e_1, e_2, e_3\}$ is spanning set of $\mathbb{R}^3$

Ex:- ~~Does~~ Is $\{e_1, e_2\}$ a spanning set of $\mathbb{R}^2$?

Ans: Yes.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$??

$c_1 = x$
$c_2 = y$

$\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \end{array} \right]$

if $y = 0$ $c_2 = 0$
$y \neq 0$ $c_2 = y$
↓
consistent always. ⇒ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ non singular.

⇒ $\{e_1, e_2\}$ is a spanning set

In general, $\{e_1, e_2, \dots, e_n\}$ is a spanning set of $\mathbb{R}^n$
↑ theorem

---

② Does $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ span $\mathbb{R}^3$?

Answer let $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}1\\2\\2\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Augmented matrix

$c_1 + c_2 = x$
$2c_1 + 2c_2 = y$
$3c_1 + 2c_2\ c_3 = z$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 2 & 2 & 0 & y \\ 3 & 2 & 1 & z \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 0 & 0 & y-2x \\ 0 & -1 & 1 & -3x+z \end{array}\right] \begin{array}{l} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc} 1 & 1 & 0 & x \\ 0 & 1 & -1 & 3x-z \\ 0 & 0 & 0 & y-2x \end{array}\right]$$

consistent if $y = 2x$
→ the system is not always consistent.
⇒ $\{v_1, v_2, v_3\}$ is not a spanning set.

Ex ③ Is $V_1 = x$, $V_2 = 1$ $V_3 = 2x-1$ spanning set for P_3?

let $F \in P_3$

$$f(x) = ax^2 + bx + c = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 \quad ??$$

$$ax^2 + bx + c = \alpha_1 x + \alpha_2 + \alpha_3(2x-1)$$

x^2 terms:- $a = 0$

x terms:- $\alpha_1 + 2\alpha_3 = b$

x^0 terms:- $\alpha_2 - \alpha_3 = c$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b \\ 0 & 1 & -1 & c \\ 0 & 0 & 0 & a \end{array}\right]$$

consistent if $a = 0$

It's not a spanning set.

Ex: $\{1, 3x\}$ is a spanning set for P_2 T or F ?

Ans (T)

let $f = ax+b$

$ax+b = \alpha_1(1) + \alpha_2(3x)$?

x: $3\alpha_2 = a$
$\alpha_1 = b$

$$\left[\begin{array}{cc|c} 1 & 0 & b \\ 0 & 3 & a \end{array}\right]$$

since $\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ system consistent (always)

$\Rightarrow$ spanning set.

In general, $\{1, x, x^2, \dots, x^{n-1}\}$ is a spanning set for P_n

Ex:- for $\mathbb{R}^{2\times2}$ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem:- let V be a vector space & let $v_1, \ldots, v_n \in V$
then $S = \text{span}(v_1, \ldots, v_n)$ is a subspace of V

Proof:- ① $S \neq \phi$

since $\vec{0} = 0 \cdot v_1 + 0 \cdot v_2 - - + 0 v_n$

② let $x, y \in S$

$x = \alpha_1 v_1 \cdots + \alpha_n v_n$

$y = \beta_1 v_1 \_\_ + \beta_n$

$(x+y) = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_n + \beta_n) v_n$

$x+y = \gamma_1 v_1 \cdots + \sigma_n v_n \, , \quad \gamma_i = \alpha_i + \beta_i$

$\Rightarrow x+y \in S$

③ x ~~y~~ $\in S$ & let α be scalar

$\alpha x = \alpha(c_1 v_1 + - - + c_n v_n)$

$= (\alpha c_1) v_1 + - - - + (\alpha_n c_n) v_n$

$\Rightarrow \alpha x \in S$

q.e.d.

* Linear System Revisited

$Ax = b$

Theorem:- let A be an $m \times n$ matrix, let $Ax = b$ be consistent with x_0 a solution. Then y is a solution of $Ax = b$ iff $y = x_0 + z$, $\underbrace{z \in N(A)}_{Az = 0}$

proof ($\Rightarrow$) Given $Ax_0 = b$

$Ay = b$

We need to prove

$y = x_0 + z$, $Az = 0$

Indeed, $Ax_0 = b$ & $Ay = b$

$Ay = Ax_0$

$A(y - x_0) = 0$

$y - x_0 \in N(A)$

$y - x_0 = z$, $z \in N(A)$

$\Rightarrow y = x_0 + z$, $z \in N(A)$

conversely, ($\Leftarrow$) suppose that

$y = x_0 + z$ $Az = 0$

We need to prove

$Ay = b$

Indeed, Ay ~~=~~

$A(x_0 + z)$

$= Ax_0 + Az$

$= b + 0 = b$

q.e.d

3.2

Theorem:- A is nonsingular iff $N(A) = \{0\}$

$Ax = 0$

Proof: Suppose the A is non singular. we need to show $N(A) = \{0\}$

$\{0\} \subseteq N(A)$ since ~~N(A) subspace~~. $A0 = 0$

let $x \in N(A)$ $\quad Ax = 0$

$$\Rightarrow A^{-1}Ax = A^{-1}0$$

$$\boxed{x = 0} \in \{0\}$$

$$\Rightarrow N(A) \subseteq \{0\}$$

Therefore $\Rightarrow N(A) = \{0\}$ set theory.

($\Leftarrow$) Suppose that $N(A) = \{0\}$ we want to show that A is nonsingular, Indeed,

Suppose not $\rightarrow$ A singular.

$\rightarrow$ $Ax = 0$ has nontrivial Solution

$\Rightarrow N(A) \neq \{0\}$

which is a contradiction.

Hence, a is nonsingular.

3.3 Linear Independence

Df:- The vector $V_1, V_2, \ldots, V_n$ in a Vector space V are said to be linear independent (L.I)

if $C_1V_1 + C_2V_2 + \cdots + C_nV_n = 0$

$$\Rightarrow C_1 = C_2 = \cdots = C_n = 0$$

The vector $V_1, \ldots, V_n$ are linearly dependent (L.D) if there exists scalars $C_1, C_2, \ldots, C_n$ not all zero, such that

$$C_1V_1 + C_2V_2 + \cdots + C_nV_n = 0$$

Ex.- Are $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ lin. indep. or lin dep ?

Sol let $C_1V_1 + C_2V_2 = \begin{pmatrix}0\\0\end{pmatrix}$

$$C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_1 + C_2 = 0$$
$$-1(C_1 + 2C_2 = 0)$$
$$-C_2 = 0 \rightarrow C_2 = 0$$
$$C_1 = 0$$

$\therefore$ linearly independent (L.I)

Ex:- $\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$ L.D in $\mathbb{R}^3$?

let $c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow c_1 = 0$

$c_2 = 0$

$c_1 = 0$ $\Rightarrow c_1 = c_2 = 0$ (L.I)

Theorem 3.3.1 : let $x_1, x_2, \ldots x_n$ be vectors in $\mathbb{R}^n$ & let $X = (x_1, x_2, \ldots, x_n)$ then tere vectors $\{x_1, \ldots x_n\}$ L.D iff X is singular ($|X| = 0$)

Remark:- we use this theorem for squar matrix X.

Ex:- $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ in $\mathbb{R}^2$

$X = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ $\quad |X| = 1(2) - (1)(1)$

$= 1 \neq 0$

X is nonsingular

$\xrightarrow{\text{Thm 3.3.1}} \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ are linearly independent.

Ex:- Are $\{P_1(x), P_2(x), P_3(x)\}$ L.I?

where

$P_1(x) = 2x^2+x+8$

$P_2(x) = x^2+8x+7$

$P_3(x) = x^2-2x+3$

let $c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) = 0$

$c_1(2x^2+x+8)+c_2(x^2+8x+7)+c_3(x^2-2x+3) = 0$

$x^2: \; 2c_1+c_2+c_3=0$

$x: \; c_1+8c_2-2c_3=0$

$x^0: \; 8c_1+7c_2+3c_2=0$

$$X = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & -2 \\ 8 & 7 & 3 \end{bmatrix}$$

← use the polynomial directly, notice the order, (rows or columns)

$$|X| = 2\begin{vmatrix} 8 & -2 \\ 7 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 8 \\ 8 & 7 \end{vmatrix}$$

$= 0$

$\Rightarrow$ X is singular $\Rightarrow$ L.D

Ex: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ L.D?

Sol $\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \neq 0 \rightarrow$ L.I

Rmk: ① if $v_1, \dots, v_n$ span V and one of these vectors can be written as a linear combination of the other $(n-1)$ vectors, then those $(n-1)$ vectors span V

Ex: let $\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\} = S$ $\qquad S = \text{span} \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Notice that $v_3 = v_1 + v_2$

$v_1 = v_3 - v_2$

$v_2 = v_3 - v_1$

$S = \text{span}\{v_1, v_2\} \Rightarrow$ check for $v_2 = \alpha v_1$?

or use determinant.

$S = \text{span}\{v_1, v_3\}$

$S = \text{span}\{v_2, v_3\}$

RMK② Given n vectors $\{V_1, \ldots, V_n\}$ it is possible to write one of the vectors as a linear combination of the other $(n-1)$ vectors iff $\{V_1, \ldots, V_n\}$ are L.D.

EX:-

$$\left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}, \underset{V_3}{\begin{pmatrix} 3 \\ 3 \end{pmatrix}} \right\} = S$$

is linearly dependent.

since $V_3 = V_1 + V_2$

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

$$V_2 = 2V_1$$

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$S = \text{span}\{V_1\}$

or $\text{span}\{V_2\}$

or $\text{span}\{V_3\}$

Ex: $S = \{\underset{f}{1}, \underset{g}{x}, \underset{h}{2-5x}\}$

Notice $h = 2f - 5g$ L.D

Span $\{1, x\} = S$

span $\{1, 2-5x\} = S$

$S =$ span $\{x, 2-5x\}$ L.I

Thm 3.3.2 let $v_1, \dots, v_n$ be vectors in V. Then $v \in \text{span}(v_1, \dots, v_n)$ can be written uniquely as $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ iff $\{v_1, \dots, v_n\}$ are L.I

Ex: $S = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}} \right\}$ is L.I

is $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in$ span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$?

Yes, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ unique : since

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$ L.I (Thm 3.3.2)

3.3 Continue.

Recall, $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$ L.I

if not all c_i is zero L.D

The vector space $C^{n-1}[a,b]$

Def: let $f_1, f_2, \dots, f_n \in C^{n-1}[a,b]$ we define $W(f_1, \dots, f_n)$ on $[a,b]$ by

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & \dots & \dots & f_n^{(n-1)} \end{vmatrix}$$

$W(f_1, \dots, f_n)$ is called the Wronskian of $f_1, f_2, \dots, f_n$

Ex:- Find $W(1, x)$:n ← اقترانين

Soln $W(1,x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$

Ex:- $W(1, x, x^2) = $??

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2$$

Ex:- $W(1,x,x^3) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x$

Theorem:- let $f_1, f_2, \dots, f_n \in C^{n-1}[a,b]$
if $\exists\ x_0 \in [a,b]$ such that $W(f_1,\dots,f_n)(x_0) \neq 0$
then $\{f_1, f_2, \dots f_n\}$ are linearly indep. (L.I)

Contrapositive:- if $f_1, f_2, \dots, f_n$ are linearly dependent, then
$W(f_1,\dots,f_n)(x) = 0$, for all $x \in [a,b]$

be careful ⚠ :- If $W(f_1,f_2,\dots,f_n)(x) = 0$
$\forall\ x \in [a,b]$, we cannot conclude, by this theorem.
In this case, we use the defn.

Ex: Are $\{1, x, x^3\}$ L.D or L.I?

Soln $W(1,x,x^3) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x$

$W(1,x,x^3)(2019) = 6(2019) \neq 0$
$\Rightarrow$ linearly independent.

Ex:- $\{e^{x}, e^{-x}\}$ L.D or L.I?

$$W(e^{x}, e^{-x}) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1-1 = \underline{\underline{-2}} \neq 0$$

$$W(e^{x}, e^{-x})(1) = -2 \neq 0 \rightarrow \text{L.I}$$

Ex: $W(x^2, x|x|)$ on $[-1,1]$ L.D or L.I?

$$|f|' = \frac{f'}{|f|} f.$$

or
$$x|x| = \begin{cases} x^2 & 0 \leq x < 1 \\ -x^2 & -1 \leq x < 0 \end{cases}$$

$$f' = \begin{cases} 2x, & 0 < x < 1 \\ -2x, & -1 < x < 0 \end{cases}$$

$$f' = 2|x|$$

$$W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0 \text{ for all } x$$

we cannot conclude by the latter.

$\Rightarrow$ We use the def:-
let $c_1 x^2 + c_2 x|x| = 0$, for all $x \in [-1,1]$

take some

$\boxed{x=-1}$ $c_1 - c_2 = 0$

$\boxed{x=1}$ $c_1 + c_2 = 0$

$2c_1 = 0 \rightarrow c_1 = 0 \quad c_2 = 0 \rightarrow$ L.I by def
?

Ex: $\{x^2, x|x|\}$ on $[0,1]$ = $\{x^2, x^2\}$ lin. dep.

Ex: $\{x^2, x|x|\}$ on $[-1,0]$ = $\{x^2, -x^2\}$ lin. dep.

Ex: $\{\underset{f_1}{1}, \underset{f_2}{\cos^2 x}, \underset{f_3}{\sin^2 x}, \underset{f_4}{x}\}$

Notice $f_1 = f_2 + f_3 + 0(x) \Rightarrow$ L.D

Ex: $\{\cos 2x, \cos^4 x - \sin^4 x, 1\}$

Notice that $g = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$= \cos 2x (1)$

$= f$

$g = f + (0)h$ lin. dep.

Ex: $\{\underset{f}{\cosh x}, \underset{g}{e^x}, \underset{h}{e^{-x}}\}$

$f = \cosh x = \frac{1}{2} e^x + \frac{1}{2} e^{-x}$

$= \frac{1}{2} g + \frac{1}{2} h$ L.D.

Ex: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\quad v_1 \quad\quad v_2 \quad\quad v_3$

lin. dep since $v_1 + v_2 = v_3$

3.4 Basis & Dimension

Def: The vectors $v_1, \dots, v_n$ form a basis for a vector space V iff (i) $v_1, v_2, \dots, v_n$ are lin. indep.

(ii) $v_1, \dots, v_n$ span V

[i.e $v_1, \dots, v_n$ is a spanning set)

Ex: $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$ and is called standard basis since

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ lin. indep.

(ii) let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \overset{?}{=} c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c_1 = a$

$c_2 = b$ $\rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$ consistent always $\rightarrow$ spanning set.

$c_3 = c$

$\therefore \{e_1, e_2, e_3\}$ is a basis

Ex: $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

Ex: $\{e_1, e_2, \ldots, e_n\}$ is standard basis for $\mathbb{R}^n$

Ex: $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \right\}$ is not a basis for $\mathbb{R}^2$

sine the set is linearly dependent $\begin{vmatrix}1 & 2\\1 & 2\end{vmatrix} = 0$

$\mathbb{R}^n \swarrow$ true

or $V_2 = 2V_1 \rightarrow$ not a basi's.

- Dimension = # of elements in a basis (dim V)

  In the previous example

  $\dim \mathbb{R}^3 = 3$

  $\dim \mathbb{R}^2 = 2$

  $\dim \mathbb{R} = 1$

in Geneneial $\rightarrow$ $\dim \mathbb{R}^n = n \rightarrow$ finite dimensional.

Ex: $P_n = \{ f(x) = a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$P_3 = \{ f = a x^2 + bx + c \}$

standard basis $= \{1, x, x^2\}$ $\dim P_3 = 3$

$\dim P_n = n \rightarrow$ finite dimensional.

standard Basis for P_n is $\{1, x, x^2, \ldots, x^{n-1}\}$

Ex: $\mathbb{R}^{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a, b, c, d \in \mathbb{R} \right\}$

Standard Basis for $\mathbb{R}^{2\times2}$ $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\dim \mathbb{R}^{2\times2} = 2\times2 = (2)(2) = \underline{4}.$

In general, $\dim \mathbb{R}^{m\times n} = (m\times n) \rightarrow$ finite dimensional

Ex: $\dim \mathbb{R}^{4\times5} = 20$

Recall, In $\mathbb{R}^n$, standard basis is $\{e_1, e_2, \ldots, e_n\}$

① $\dim \mathbb{R}^n = n$

② $P_n = \{ f(x) : a_{n-1}x^{n-1} + \cdots + a_0 \}$

standard basis is $\{1, x, x^2, \ldots, x^{n-1}\}$

$\dim P_n = n$

③ $\mathbb{R}^{m\times n} = \{A_{m\times n}\}$

$\dim \mathbb{R}^{m\times n} = mn$

ex: $\dim \mathbb{R}^{4\times5} = 20$

Standard Basis for $\mathbb{R}^{3\times2}$ is

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

④ dim $\underset{\downarrow}{C^n}[a,b]$ = infinite.
doesn't have a basis

Any set containing 0 is linearly dependent → not basis $\{0, v_1, v_2 \ldots\}$
$0 = 0v_1 + 0v_2 + \ldots$

Ex:- dim $\mathbb{R}$ = 1

Ex:- dim $\{0\} = 0$ logically. $\{0\}$ → not a basis.

Ex: Find a basis and dimension for $N(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

soln:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] -2R_1+R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] -R_2$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

x_1, x_2 are leading
$x_3 = t$ is free
$x_1 = x_3 = t$
$x_2 = -2x_3 = -2t$

$$N(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

& it's linearly independent

Note: every set of one element is L.I if not zero element.

Notice that $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$ is lin. indep

$\Rightarrow$ a basis for $N(A) = \left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$

$\dim N(A) = 1 \Rightarrow$ # of free variables.

<u>**Ex:**</u> find a basis & dim

of $S = \left\{\begin{pmatrix}a-b+c\\2b-3c\\4a+2c\end{pmatrix} : a, b, c \in \mathbb{R}\right\}$

maximum 3 is the dimension if not linearly dep --

$\downarrow$ # of free var. check.

$x \in S \Rightarrow x = a\begin{pmatrix}1\\0\\4\end{pmatrix} + b\begin{pmatrix}-1\\2\\0\end{pmatrix} + c\begin{pmatrix}1\\-3\\2\end{pmatrix}$

$\therefore \left\{\begin{pmatrix}1\\0\\4\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\2\end{pmatrix}\right\}$ is spanning set

- Lin. Indep?

$$\begin{vmatrix}1 & -1 & 1\\0 & 2 & 3\\4 & 0 & 2\end{vmatrix} = 24 \neq 0 \quad \therefore \text{lin. indep.}$$

$$\therefore \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\} \text{ is a basis}$$

$\dim S = 3$

Ex: Find a basis & dim

of $S = \left\{ (a+3b+c,\ 2a+6b,\ c)^T,\ a, b, c \in \mathbb{R} \right\}$

Sol let $X \in S$

$$X = a\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \right\}$$ is lin. dep since

$v_1 = \frac{1}{3} v_2$

Remove v_1 or v_2 say v_1

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$ lin. indep (not multiples of each others) or solve it.

$\dim S = 2$

RMK $\dim N(A) = \text{nullity}(A)$

Ex: $S = \{q(x) \in P_3 : q''(x) = 0\}$

$q \in S = q(x) = ax^2 + bx + c$

$q'(x) = 2ax + b$

$q''(x) = 2a = 0 \Rightarrow \boxed{a = 0}$

Therefore, $q(x) = bx + c.1 = \text{span}\{x, 1\}$

- $\{x, 1\}$ linearly independent

$\Rightarrow \{x, 1\}$ is a basis for $S \Rightarrow \boxed{\dim S = 2}$

Ex $S = \{p \in P_3 : p(0) = 0\}$

$p(x) \in S \Rightarrow p(x) = ax^2 + bx + c$

$p(0) \Rightarrow \boxed{c = 0}$

$\therefore p(x) = ax^2 + bx$

$= \text{span}\{x^2, x\}$ linearly independent

$\Rightarrow \{x^2, x\}$ is a basis for S

$\dim S = 2$

Ex: $S = \{p \in P_3 : p(x) = ax^2 + bx + 2a + 3b\}$

Sdn $q \in S$

$$q(x) = a(x^2+2) + b(x+3)$$

$$= \text{span}\underbrace{\{x^2+2,\ x+3\}}_{\text{basis}} \leftarrow \text{linearly independent}$$

$\dim S = 2$

Ex: $S = \{A \in \mathbb{R}^{2\times 2} : A^T = A\}$

$$A \in S \Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = A \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \boxed{b = c}$$

$$\therefore A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \text{span}\left\{ \underset{V_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{V_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{V_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\} \quad \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

lin. indep.

basis $\therefore$ dim $= 3$

Theorem 3.4.1 let $\{w_1, w_2, \ldots, w_n\}$ be a spanning set for V then $\{w_1, \ldots, w_k\}$, $k > n$ are linearly dep & Hence not a basis.

Ex:- $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\}$ is spanning set for $\mathbb{R}^2$

then $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix} \right\}$ is lin. dep.

Corollary 3.4.2 If $\{v_1, \ldots, v_n\}$ & $\{w_1, \ldots, w_k\}$ are two bases for V, then $n = k$ (any two bases have same dimension)

Ex: $\overset{S_1}{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}$ is a basis for $\mathbb{R}^2$

Also $\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right\}^{S_2}$ " " $\mathbb{R}^2$

$\Rightarrow \dim S_1 = \dim S_2 = 2 \quad (n = k)$

$\boxed{3.4}$ Continue

<u>Thm</u>: let V be a vector space with $\dim V = n > 0$. Then the following are equivalent:-

not trivial subspace
zero subspace

(i) $\{v_1, \dots, v_n\}$ is a basis.
(ii) $\{v_1, \dots, v_n\}$ span V.
(iii) $\{v_1, \dots, v_n\}$ lin. indep.

condition (implicit):-
$\dim V = n = \#$ of vectors.

<u>Ex</u>:- $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right\}$ in $\mathbb{R}^2$ theorem can be applied $2 = 2$ ✓

<u>Sol</u> $\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} = 6 \neq 0$ lin. independent.

Thm $\Rightarrow$ Basis.

<u>Ex</u>:- $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \right\}$ in $\mathbb{R}^3$ thm can be applied $3 = 3$ ✓

<u>Soln</u> $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 20 \neq 0$

$\rightarrow$ lin. indep $\xRightarrow{\text{Thm}}$ Basis.

<u>Summary</u>

* let V be a vector space with $\dim V = n > 0$ Then

① A set $\{v_1, \dots, v_k\}$, $k < n$ $\rightarrow$ cannot be span V $\rightarrow$ not a basis.
② A set $\{v_1, \dots, v_k\}$ $k > n$ $\rightarrow$ Linearly Dependent $\rightarrow$ not a basis.
③ A set $\{v_1, \dots, v_k\}$ $k = n$ are lin indep <u>or</u> Span V, then $\{v_1, \dots, v_k\}$ is a basis.
[illegible]
④ A <u>spanning set</u> of $v_1, \dots v_k$ $k > n$ <u>can</u> be reduced (paredown) to a basis for V.
[illegible]
⑤ A <u>linearly indep</u> set $v_1, \dots v_k$ $k < n$ can be extended to a basis for V.

Ex: $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$ is not a basis for $\mathbb{R}^2$

$\downarrow$

linearly independent (one element $\neq 0$)

Add $\begin{pmatrix}0\\1\end{pmatrix}$ $\quad \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1 \neq 0$ lin. indep

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ basis

Also

Add $\begin{pmatrix}5\\2\end{pmatrix}$ $\quad \begin{vmatrix}1 & 5\\0 & 2\end{vmatrix} = 2 \neq 0$

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}\right\}$ basis.

Ex:- $\left\{\underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\5\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}6\\7\end{pmatrix}}\right\}$ spanning set

is lin dep (not basis)

$\begin{vmatrix}1 & 0\\0 & 5\end{vmatrix} = 5 \neq 0$

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\5\end{pmatrix}\right\}$ basis.

Q10) $X_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}, \quad X_2 = \begin{pmatrix}2\\5\\3\end{pmatrix}, \quad X_3 = \begin{pmatrix}1\\3\\2\end{pmatrix}$

$X_4 = \begin{pmatrix}2\\7\\4\end{pmatrix}, X_5 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ span $\mathbb{R}^3$. (Given).

Pare down $\{X_1, X_2, X_3, X_4, X_5\}$ to form a basis for $\mathbb{R}^3$?

Soln We only need 3 vectors:-

$$\begin{vmatrix}1 & 2 & 1\\3 & 7 & 1\\2 & 4 & 0\end{vmatrix} = 1\begin{vmatrix}7 & 1\\4 & 0\end{vmatrix} - 2\begin{vmatrix}3 & 1\\2 & 0\end{vmatrix} + 1\begin{vmatrix}3 & 7\\2 & 4\end{vmatrix} = -4 + 4 + -2 = \boxed{-2} \neq 0$$

$\{x_3, x_4, x_5\}$ a basis for $\mathbb{R}^3$.

$\boxed{8.5}$ Change of Basis.

Def:- let V be a vector space & let $E = \{V_1, \dots, V_n\}$ be a basis of V, then $\vec{V} \in V$ can be written uniquely

$$\vec{V} = \alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_n V_n$$

where $\alpha_1, \dots, \alpha_n$ are scalars.

The vector $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ in $\mathbb{R}^n$ is called the coordinate of $\vec{v}$ with respect to a basis E & is denoted by:-

$$[V]_E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{or } V_E$$

Ex:

let $V = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$

& let $E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

find $[V]_E$

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$2 = \alpha_1$ $\Rightarrow$ $[V]_E = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = V$

$5 = \alpha_2$

Remark:- if E is standard Basis $\rightarrow$ $[V]_E = V$

not a standard basis

Ex: $V = \begin{pmatrix}5\\6\end{pmatrix} \in \mathbb{R}^2$, $E = \overbrace{\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}}$ ⚠ order

find $[V]_E$

$$\begin{pmatrix}5\\6\end{pmatrix} = \alpha_1\begin{pmatrix}0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{matrix}\alpha_2 = 5\\ \alpha_1 = 6\end{matrix} \Rightarrow [V]_E = \begin{pmatrix}6\\5\end{pmatrix}$$

Ex: let $E = \left\{\underset{v_1}{\begin{pmatrix}3\\2\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\1\end{pmatrix}}\right\}$ be a basis for $\mathbb{R}^2$

let $x = \begin{bmatrix}7\\4\end{bmatrix}$

find $[x]_E$

Sol let $x = \alpha_1 V_1 + \alpha_2 V_2$

$$\begin{pmatrix}7\\4\end{pmatrix} = \alpha_1\begin{pmatrix}3\\2\end{pmatrix} + \alpha_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$7 = 3\alpha_1 + \alpha_2$$
$$4 = 2\alpha_1 + \alpha_2$$

$$\underbrace{\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}}_{U}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}7\\4\end{bmatrix}$$

has a unique soln when A is nonsingular.

$$\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = U^{-1}\begin{bmatrix}7\\4\end{bmatrix}$$

$$[x]_E = \begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}7\\4\end{bmatrix}$$

$[x]_E = U^{-1} \cdot X$

$$[x]_E = \frac{1}{3-2}\begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}\begin{bmatrix}7\\4\end{bmatrix} = \begin{bmatrix}7-4\\-14+12\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$$

$$[x]_E = \begin{pmatrix}3\\-2\end{pmatrix}$$

Def: U is called the transition matrix from $E = \{u_1, v_2\}$ to the standard basis $\{e_1, e_2\}$

That is

$$\begin{tikzcd}
{[u_1, v_2]} \arrow[r, "U"] \arrow[dr, "V^{-1}U"'] & {[e_1, e_2]} \arrow[l, dashed, bend left, "U^{-1}"] \arrow[d, "V^{-1}"] \\
 & {[v_1, v_2]}
\end{tikzcd}$$

$S = V^{-1}U$ transition matrix from U to V.

(from one order basis $[u_1, v_2]$
to another order basis $[v_1, v_2]$)

Ex: let $E = \left\{ \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \end{pmatrix} \right\}$ &

$F = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be two bases in $\mathbb{R}^2$

Find the transition matrix from E to F

Soln

$$S_{E \to F} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix} \qquad S_{E \to F} \neq S_{F \to E}$$

RMK $[x]_F = S_{E \to F} [x]_E$

→

11

3-12

-2×9

Ex: $E = [3x+6,\ 9]$

$F = [2x+1,\ x-4]$

$$\left(\begin{array}{cc|cc} 1 & -4 & 6 & 9 \\ 2 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -4 & 6 & 9 \\ 0 & 9 & -9 & -18 \end{array}\right)$$

Soln: $\hat{\text{for}}$ P_2.

(a) Find the transition matrix $S_{E\to F}$

(b) use (a) to Find the coordinate $[3x+15]_F$

$$\left[\begin{array}{cc|cc} 1 & -4 & 6 & 9 \\ 0 & 1 & -1 & -2 \end{array}\right] \downarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

resit

Rmk
يدراب و constant وبطلع

(a) $S_{E\to F} = F^{-1}E$

$$= \begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \quad S'_{E\to F}$$

(b) $[3x+15]_F = \overset{S_{E\to F}}{\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}} [3x+15]_E$ → by inspection $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ بالنظر

$$= \begin{bmatrix} 2+1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$\boxed{3.6}$ Row space and column space.

Def: If A is $m\times n$ matrix, then

① The subspace of $\mathbb{F}^{1\times n}$ spanned by the row vectors of A is called row space of A denoted by $\mathbb{R}(A)$

② The subspace of $\mathbb{F}^{m}$ spanned by the column vectors of A is called the column space of A is called the column space of A denoted by $C(A)$

Ex: let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ find $\mathbb{R}(A)$ & $C(A)$

soln $\mathbb{R}(A) = \text{span}\{(1,0,0), (0,1,0)\}$ subspace of $\mathbb{F}^3$

$= \{V : V = \alpha(1,0,0) + \beta(0,1,0)\}$ dim $= 2$

$= \{(\alpha, \beta, 0) : \alpha, \beta \text{ scalars}\}$

$C(A) = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$

$= \left\{V : V = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} + \gamma\begin{pmatrix}0\\0\end{pmatrix}\right\}$

$= \left\{\begin{pmatrix}\alpha\\\beta\end{pmatrix} : \alpha, \beta \text{ scalars}\right\}$ subspace of $\mathbb{F}^2$

$= \mathbb{F}^2$ dim $= 2$.

Theorem : Two row equivalent matrices have the same row space.

Def: ① the rank of matrix $A_{m\times n}$ denoted by $\operatorname{rank}(A)$ is $\operatorname{rank}(A) = \dim \mathbb{R}(A) = \dim C(A)$

② $\dim N(A) = \text{nullity}(A)$

③ **Theorem**: (Rank-nullity Thm)

$$\operatorname{rank}(A) + \text{nullity}(A) = \overset{\#\text{ of columns.}}{n},$$

where $A_{m\times n}$

Theorem: if A is $m\times n$ matrix, then $\dim C(A) = \dim \mathbb{R}(A) = \operatorname{rank}(A)$

if $\mathbb{R}(A) = C(B) \longrightarrow \dim($?

Ex: Find a basis for $R(A)$ & $c(A)$ & $N(A)$ Then find rank(A) and nullity(A). Where $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}$

Sol 1st $A \rightsquigarrow$ REF or RREF say U

$$A \overset{\text{rref}}{\underset{\text{ref}}{\rightsquigarrow}} U$$

trivial subspaces ↓ فقط $\{0\}$ element.

Ther non zero rows in U form a basis for $R(A)$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & -2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & -2 & -10 \end{bmatrix}$$

$$\text{RREF} \xrightarrow{\text{أصل}} \begin{array}{c} * \\ * \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

two rows → rank = 2
↓
nonzero

$*$ a Basis for $R(A)$: $\{(1\,0\,13), (0\,1\,5)\}$ $\dim R(A) = 2$
$= \text{rank}(A)$

$N(A) = ?$ $AX = 0 \to UX = 0$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 13 & 0 \\ 0 & \textcircled{1} & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 leadings
$x_3 = t$ free

$x_1 = -13t$
$x_2 = -5t$
$x_3 = t$

$$N(A) = \left\{ \begin{pmatrix} -13t \\ -5t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} -13 \\ -5 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

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# of free variables

شرط اي ضلط تكون L.I

a basis for $N(A)$ is $\left\{ \begin{pmatrix} -13 \\ -5 \\ 1 \end{pmatrix} \right\}$ $\dim N(A) = 1$ = nullity of A

Notice that rank A + nullity A => $2 + 1 = 3 = n$

عدد الاعمدة.

a basis for $C(A)$ → the corresponding ~~rows~~ columns that have leading 1

a basis for $C(A)$ is $\{a_1, a_2\}$

$$= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ -4 \end{pmatrix} \right\} \quad \dim C(A) = 2 = \dim R(A)$$

Remark: the columns of A that correspond to the leading 1's in U will form a basis for C(A).

* Dependency Relation

In the last example
$$A = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \end{array}$$

$$U = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} ① & 0 & 13 \\ 0 & ① & 5 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad u_3 = 13u_1 + 5u_2$$

$$\Downarrow$$

$$a_3 = 13a_1 + 5a_2$$

Dependency Relation

$a_3 = 13a_1 + 5a_2$

Ex:- $A = \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \hline \end{array}$ $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$ find a basis for $R(A)$, $C(A)$, rank A, nullity A & dependency relation.

"علاقة بين أعمدة A"

Soln: $U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ \textcircled{1} & 2 & 0 & 3 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

[illegible]

a basis for $R(A)$ is $\{(1, 2, 0, 3), (0, 0, 1, 2)\}$

$\dim R(A) = 2 = \text{rank } A$

A basis for $C(A)$ is $\{a_1, a_3\}$

$= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \right\}$ $\dim C(A) = 2 = \dim R(A)$

dependency relation

$u_2 = 2u_1 + 0u_3 \rightarrow u_2 = 2u_1 \rightarrow \boxed{a_2 = 2a_1}$

$u_4 = 3u_1 + 2u_3 \rightarrow \boxed{a_4 = 3a_1 + 2a_3}$

Thm A linear system $Ax = b$ is consistent if and only if b is in the column space of A

Thm A linear system $Ax = 0$ will have only the trivial solution $x = 0$ if and only if the column vectors of A are linearly independent.

Thm A $m \times n$ matrix. $Ax = b$ is consistent $\forall b \in \mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$. The system $Ax = b$ has at most one solution $\forall b \in \mathbb{R}^m$ iff the column vectors of A are linearly indep.

Thm System of rows of square matrix are L.I iff the determintant of the matrix is not equal to zero.

Thm: An $n \times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for $\mathbb{R}^n$.

* if A is $m \times n$ matrix & the columns of A are L.D then $m \leq n$

Ch.4 Linear Transformation

4.1 Definition & Examples

Def A mapping $L: V \to W$ where V & W are vector spaces is said to be linear transformation (L.T) if:-

(1) $L(V_1+V_2) = L(V_1) + L(V_2)$ for all $V_1, V_2 \in V$

(2) $L(\alpha V) = \alpha L(V)$ where α is scalar & $V \in V$

Rmk: If $V = W$, the $L: V \to V$ is called linear operator

Ex: Is $L: [a,b] \to \mathbb{R}$ defined as $L(f(x)) = \int_a^b f(x)\,dx$ a L.T?

Ans (i) $L(f(x)+g(x)) \overset{?}{=} L(f(x)) + L(g(x))$

Indeed, $L(f(x)+g(x)) = \int_a^b (f(x)+g(x))\,dx$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$= L(f(x)) + L(g(x))$$

(ii) let α be scalar

$L(\alpha f(x)) \overset{?}{=} \alpha L(f(x))$

$L(\alpha f(x)) = \int_a^b \alpha f(x)\,dx$

$$= \alpha \int_a^b f(x)\,dx$$

$$= \alpha L(f(x)) \quad \therefore L \text{ is a L.T}$$

Ex2 Is $L: C^1[a,b] \to C[a,b]$

$L(f(x)) = f'(x)$ a L.T ?

Sol (i) $L(f(x)+g(x)) = (f(x)+g(x))'$

$= f'(x) + g'(x)$

$= L(f(x)) + L(g(x))$

(ii) Let α be scalar

$L(\alpha f(x)) = (\alpha f(x))'$

$= \alpha f'(x) = \alpha L(f(x))$

$\therefore L$ is a L.T

Ex) $L: \mathbb{R}^2 \to \mathbb{R}^2$

$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

i. $L\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) \overset{?}{=} L\begin{pmatrix} a \\ b \end{pmatrix} + L\begin{pmatrix} c \\ d \end{pmatrix}$

Now $L\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = L\begin{pmatrix} a+c \\ b+d \end{pmatrix}$

$= \begin{pmatrix} 3a+3c \\ 3b+3d \end{pmatrix}$

$= \begin{pmatrix} 3a \\ 3b \end{pmatrix} + \begin{pmatrix} 3c \\ 3d \end{pmatrix}$

$= L\begin{pmatrix} a \\ b \end{pmatrix} + L\begin{pmatrix} c \\ a \end{pmatrix}$

ii. Let α be scalar

$$L\left(\alpha\begin{pmatrix}a\\b\end{pmatrix}\right) \overset{?}{=} \alpha L\begin{pmatrix}a\\b\end{pmatrix}$$

Now $L\left(\alpha\begin{pmatrix}a\\b\end{pmatrix}\right) = L\begin{pmatrix}\alpha a\\ \alpha b\end{pmatrix}$

$$= \begin{pmatrix}3\alpha a\\3\alpha b\end{pmatrix}$$

$$= \alpha\begin{pmatrix}3a\\3b\end{pmatrix}$$

$$= \alpha L\left(\begin{pmatrix}a\\b\end{pmatrix}\right)$$

$\therefore$ L is a L.T

Ex4 $L: P_2 \rightarrow P_3$

$$L(p(x)) = p(x) + x^2$$

sol L is not a L.T

$L(p+q)$ $L((1-x)+(x-1)) = L(0) = 0 + x^2$
$= x^2$

$L(p) + L(q) = L(1-x) + L(x-1)$
$= 1-x+x^2+x-1+x^2$
$= 2x^2 \neq L(p+q)$

Ex5 $L : \mathbb{R}^2 \to \mathbb{R}^3$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Sol not L.T

$$L\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \neq L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

since $L\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thm let V, W be vector spaces, let $L : V \to W$ be a lin. T. Then

(a) $L(0_v) = 0_w$

(b) $L(v_1 - v_2) = L(v_1 + -v_2)$
$= L(v_1) + L(-v_2)$
$= L(v_1) - L(v_2)$

(c) $L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \dots + \alpha_n L(v_n)$

Def: Let $L: V \to W$ be a L.T Then

① The kernel of L is

$$\ker(L) = \{ v \in V : L(v) = 0_W \}$$

② L is 1-1 if $L(v_1) = L(v_2) \Rightarrow v_1 = v_2$

or $\ker(L) = 0_V$

③ The image (or the range) of L is denoted by

$Imm(L)$ or $L(V)$ or
R_L

$$L(V) = \{ w \in W : w = L(v) \text{ for some } v \in V \}$$

④ If $L(V) = W$, then L is said to be onto

Thm اثبات مهم let $L: V \to W$ be a L.T. Then

a) $\ker(L)$ is a subspace of V [illegible]

b) R_L is a subspace of W

Ex: $L: P_3 \to \mathbb{R}^2$ be a L.T
where $L(p(x)) = \begin{pmatrix} p''(x) - p'(1) \\ p(0) \end{pmatrix}$

1) Find
1) $\ker(L)$ & its dimension
2) R_L & its dim
3) Is L onto?
4) Is L 1-1?
5) let $S = P_1$. Find $L(S)$

$p(x) \in P_3 \rightarrow p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b$

$p''(x) = 2a$

$p'(1) = 2a + b$

$p(0) = c$

$\therefore L(ax^2+bx+c) = \begin{pmatrix} -b \\ c \end{pmatrix}$

① +④ $\ker L = \{ p(x) \in P_3 : L(p(x)) = 0_W \}$

$= \{ p(x) \in P_3 : L(p(x)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

$L(ax^2+bx+c) = \begin{pmatrix} -b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\boxed{\begin{matrix} b = 0 \\ c = 0 \end{matrix}}$$

$p(x) = ax^2 = \text{span}\{x^2\}$

A basis for $\ker(L) = \{x^2\} \rightarrow \dim \ker(L) = 1$

Since $\ker L \neq 0$

$\Rightarrow L$ is not one to one.

+③

2) R_L ? $L(ax^2+bx+c) = \begin{pmatrix} -b \\ c \end{pmatrix}$

$= b\begin{pmatrix} -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$ Lin Indep.

$\Rightarrow \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for R_L

onto. b lie) $\boxed{\dim R_L = 2 = \dim \mathbb{R}^2}$

$\Rightarrow L$ is onto.

$\equiv$ rank nullity

$\dim \ker(L) + \dim R_L$

$= 1 + 2 = 3 = \dim P_2 \leftarrow \dim V$

دائماً صح
لل finite.

(5) $S = P_2$ find $L(P_1)$

$$L(S) = L(\alpha x + \beta)$$

$$= L(0x^2 + \alpha x + \beta)$$

$$= \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}$$

Ex: $L(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Rmk: If $L: V \to W$ is a linear transformation and $\dim V < \infty$ then

$$\dim \ker(L) + \dim R_L = \dim V$$

Ex: let $L: P_3 \to P_3$ be a L.T defined by

$L(p(x)) = x^2 p''(x) + p'(x) + p(0)$

(a) Find a basis & dimension of Ker (L)

(b) " " " of range (L)

(c) Is L one to one?

(d) Is L onto?

Soln $p(x) \in P_3 \Rightarrow p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b \Rightarrow p''(x) = 2a$

$\therefore L(ax^2 + bx + c) = x^2(2a) + 2ax + b + c$

$= 2ax^2 + 2ax + b + c$

(a) + (c) $\ker(L) = \{p(x) \in P_3 : L(p(x)) = 0\}$

$L(p(x)) = 2ax^2 + 2ax + b + c = 0$

$\Rightarrow 2a = 0 \quad a = 0$

$b + c = 0 \quad b = -c$

$\therefore p(x) = ax^2 + bx + c = 0x^2 + bx - b$

$= b(x-1) = \text{span}\{(x-1)\}$

$\dim \ker(L) = 1 \Rightarrow L$ is not one to one.

(b)+(d) $R_L = ?$

$L(p(x)) = L(ax^2 + bx + c) = 2a(x^2 + x) + (b + c)$

$= \text{span}\{x^2 + x, 1\}$

$\therefore R_L = \text{span}\{x^2 + x, 1\}$

$\therefore \dim R_L = 2$

Note R_L is a subspace of P_3 ($\dim P_3 = 3$)

$\Rightarrow R_L$ is not onto.

__Ex:__ let $L: \mathbb{R}^3 \to P_4$ be a L.T defined by

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b)x^3 + (b+c)x^2 + (a+c)x$$

(1) Find a basis & dim of $Ker(L)$?
Is L 1-1?

(2) Find a basis & dim of R_L?
Is L onto?

__Soln:__ ① $Ker(L) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0_{P_4} \right\}$

$(a+b)x^3 + (b+c)x^2 + (a+c)x = 0$

$$\Rightarrow \begin{aligned} a+b&=0 \\ b+c&=0 \\ a+c&=0 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \Rightarrow \begin{aligned} 2c &= 0 \quad \boxed{c=0} \\ b+c &= 0 \quad \boxed{b=0} \\ a+b &= 0 \quad \boxed{a=0} \end{aligned}$$

$a = b = c = 0$

$Ker(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow L$ is 1-1

a basis of $Ker(L)$ is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ & $\dim Ker(L) = 0$.

② $R_L = ?$

$$\begin{aligned} L\begin{pmatrix} a \\ b \\ c \end{pmatrix} &= (a+b)x^3 + (b+c)x^2 + (a+c)x \\ &= a(x^3+x) + b(x^3+x^2) + c(x^2+x) \\ &= span\{x^3+x,\ x^3+x^2,\ x^2+x\} \end{aligned}$$

lin. indep (check)

$\therefore$ A basis for $R_L = \{x^3+x,\ x^3+x,\ x^1+x\}$

$\dim R_L = 3 < \dim P_4 \to$ not onto.

Ex: If $T: P_2 \to P_2$ is a L.T with $T(x+1) = 2$ & $T(x-2) = -1$
Find $T(-3)$

Soln: Firstly we write $-3 = \alpha(x+1) + \beta(x-2)$

$$\Rightarrow \alpha + \beta = 0$$
$$\alpha - 2\beta = -3 \qquad \boxed{\alpha = -1} \quad \boxed{\beta = 1}$$

$$-3 = -1(x+1) + 1(x-2)$$
$$T(-3) = -1T(x+1) + 1T(x-2)$$
$$= -1(2) + 1(-1) = \underline{\underline{-3}}$$

Q4) $L: R^2 \to R^2$ is a L. operator.

$L\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$ $L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$ find $L\begin{pmatrix}7\\5\end{pmatrix} = ?$

$$\begin{pmatrix}7\\5\end{pmatrix} = \alpha\begin{pmatrix}1\\-1\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix}$$

$$\alpha + \beta = 7$$
$$-\alpha + 2\beta = 5 \qquad \beta = 4 \qquad \alpha = 3$$

$$\begin{pmatrix}7\\5\end{pmatrix} = 3\begin{pmatrix}1\\-1\end{pmatrix} + 4\begin{pmatrix}1\\2\end{pmatrix}$$

$$L\begin{pmatrix}7\\5\end{pmatrix} = 3L\begin{pmatrix}1\\-1\end{pmatrix} + 4L\begin{pmatrix}1\\2\end{pmatrix} \qquad \text{sin L is L.T}$$

$$= 3\begin{pmatrix}5\\2\end{pmatrix} + 4\begin{pmatrix}-2\\3\end{pmatrix}$$

$$= \begin{pmatrix}15\\6\end{pmatrix} + \begin{pmatrix}-8\\12\end{pmatrix} = \begin{pmatrix}7\\18\end{pmatrix}$$

Q14) $L: \mathbb{R} \to \mathbb{R}$ lin opr. let

$L(1) = a$ show that $L(x) = ax \quad \forall\, x \in \mathbb{R}$

pf: $L(x) = L(x \cdot 1) = xL(1) = xa = ax$

Q22) $L: \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix}$

Is L 1-1? onto? Justify.

Soln $\ker(L) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Now, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x = 0$

$x + y = 0 \qquad x = y = z = 0.$

$x + y + z = 0$

$\therefore \ker(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\dim \ker(L) = 0$ 1-1.

$R_L = ??$

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \operatorname{span}\underline{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

L.I check

A basis for $R_L = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim R_L = 3 = \dim \mathbb{R}^3$

$\Rightarrow$ L is onto

Thm 4.1.1 let $L: V \to W$ be a L.T. Then

(a) ker (L) is a subspace of V

(b) $L(V) = R_L$ is a subspace of W

proof: [see textbook thm 4.4.1 page 175 + 176]

Def: 6.1 Eigenvalues & Eigenvectors

Def: let A be $n\times n$ matrix. A scalar λ is an eigen value of A or characteristic value of A if $\exists$ a non zero vector V such that

$$\boxed{AV=\lambda V}$$

The vector V is an eigne vector corresponding to λ

ex: $A=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}$, $V=\begin{bmatrix}2\\1\end{bmatrix}$ $\lambda=3$

Sol $AV=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}6\\3\end{bmatrix}=3\begin{bmatrix}2\\1\end{bmatrix}=3V$

$\lambda=3$ is an eigen value of A

$V=\begin{bmatrix}2\\1\end{bmatrix}$ is the corresponding eigenvector of $\lambda=3$

Question: How to find an eigenvalue & corresponding eigen vector of a matrix A?

Ans $AV=\lambda V$ $V\neq 0$

$AV-\lambda V=0$

$(A-\lambda I)V=0$ $(A-\lambda I)V=0$ $V\neq 0$ (nontrivial)

$\Rightarrow$ $A-\lambda I$ is singular

$\boxed{\det(A-\lambda I)=0}$ characteristic equation. نطرح λ من main diagonal.

eigen vector $=N(A-\lambda I)$.

Ex: Find the eigen values & the corresponding eigen vectors of

(1) $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

Sol The characteristic eq is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-2-\lambda) - 6 = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0 \qquad \boxed{\lambda_1 = 4} \quad \boxed{\lambda_2 = -3}$$

→ eigen values.

For $\boxed{\lambda_1 = 4}$, let v_1 be an eigen vector

$N(A - \lambda_1 I)$

$$A - \lambda_1 I = A - 4I = \begin{bmatrix} 3-4 & 2 \\ 3 & -2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

$$N(A - 4I) \rightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} ① & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

دايماً
- free
infinite
many

$x_1 = 2x_2$ let $x_2 = t \Rightarrow x_1 = 2t$

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$$N(A-4I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$\therefore$ $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda_1 = 4$

Similarly

for $\boxed{\lambda_2 = -3}$ let V_2 be an eigen vector belonging to $\lambda_2 = -3$

we must find $N(A - \lambda_2 I)$

$N(A+3I) = ?$

$$A + 3I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 3 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} a & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$a + \frac{1}{3} b = 0 \qquad a = -\frac{1}{3} b$

let $b = r \rightarrow a = -\frac{1}{3} r$

$$N(A+3I) = \left\{ \begin{pmatrix} -\frac{1}{3} r \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$$

$$= \left\{ r \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} : r \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

$\therefore$ $V_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigen vector corr. to $\boxed{\lambda_2 = -3}$

Ex2: $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

Sol. $|A-\lambda I| = 0$ $\quad \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$

$$\therefore (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix} = 0$$

$$\therefore (2-\lambda)\left[(-2-\lambda)(2-\lambda)+3\right] + 3\left[(2-\lambda)-1\right] + \left[-3+2+\lambda\right] = 0$$

$$\vdots$$

$$-\lambda(\lambda-1)^2 = 0 \qquad \boxed{\lambda_1 = 0} \qquad \boxed{\lambda_2 = 1 = \lambda_3}$$

are the eigen values.

for $\boxed{\lambda_1 = 0}$ let $v_1 \sim$,

$N(A - 0I) = N(A)$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \sim \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 leading, x_3 free

$x_3 = t \rightarrow x_1 = t \qquad x_2 = x_3 = t$

$$\therefore N(A) = \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$v_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is an eigenvector. corr to $\boxed{\lambda = 0}$

for $\lambda_2 = \lambda_3 = 1$

$N(A-I) = ?$

$N(A-I) = ?$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 leading $\quad x_2 = t$, $x_3 = r$

$x_1 = 3x_2 - x_3 = 3t - r$

$$N(A-I) = \left\{ \begin{pmatrix} 3t - r \\ t \\ r \end{pmatrix} \right\} = \left\{ t \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + r \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{v_3} \right\}$$

Ex ③ $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $\quad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)(4-\lambda)(2-\lambda) = 0$$

$$\underbrace{\boxed{\lambda_1 = 2} \quad \boxed{\lambda_2 = 4} \quad \boxed{\lambda_3 = 2}}_{\Downarrow}$$

the entries of the main diagonal upper/lower /diagonal → quick.

<u>Def:</u> trace of $A_{n\times n}$

$tr(A) =$ sum of all entries on the main diagonal

<u>Ex:</u> $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ $\quad tr(A) = 3 + -2 = 1$

$\lambda_1 = 4 \quad \lambda_2 = -3$

$\lambda + \lambda_2 = 4 - 3 = 1 = tr(A)$

بعد التكرار!

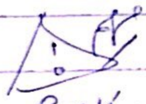
$\lambda_1 \cdot \lambda_2 = (4)(-3) = -12$

$\det A = (3)(-2) - (3)(2) = -12$

$= \lambda_1 \lambda_2$

<u>Rmk</u> ① $A_{\textcircled{n}\times\textcircled{n}}$, $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_{\textcircled{n}}$. بحسب تكرار

② $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_{\textcircled{n}}$ بحسب تكرار



eigen [illegible] v

<u>Rmk:</u> $A_{n\times n}$ matrix, λ is a scalar. Then the following are equivalent:-

(a) λ is an eigen value of A

(b) $(A - \lambda I)\, v = 0 \quad v \neq 0$

(c) $N(A - \lambda I) \neq \{\vec{0}\}$

(d) $A - \lambda I$ is singular

(e) $\det(A - \lambda I) = 0$