1.4 nonsingular matrices

$\underset{n\times n}{A}$ nonsingular $\iff \exists \underset{n\times n}{B}$ s.t

$$AB = \boxed{I = BA}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$, $\boxed{B = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}}$

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$BA = I$ $\implies$ A is nonsingular and

$A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$, and B is nonsing. $B^{-1} = A$.

* If $\underset{n\times n}{C}\ \underset{n\times n}{D} = I = DC \implies$

① C is nonsing, $C^{-1} = D$ and

② D " " , $\boxed{D^{-1} = C.}$

$\boxed{A \text{ nonsingular}}$

$\boxed{A^{-1} \text{ is nonsingular and } (A^{-1})^{-1} = A.}$

* If $A_{n\times n}$ has no inverse, we say A is singular. $\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ singular

* **Th** If $A_{n\times n}, B_{n\times n}$ are nonsingular, then $\boxed{AB}$ is nonsingular and $(AB)^{-1} = \boxed{B^{-1}A^{-1}}$

Proof: A, B nonsing. A^{-1} exists, B^{-1} exists

$$AA^{-1} = I = A^{-1}A, \quad BB^{-1} = I = B^{-1}B.$$

$$\boxed{(AB)}\boxed{B^{-1}A^{-1}} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I_n$$

ad $(B^{-1}A^{-1})(AB) = I$

So AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

* If $A_1, A_2, \cdots, A_k$, are nonsingular $n\times n$-mat then $A_1A_2\cdots A_k$ is nonsingular and

$$((A_1A_2\cdots)A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

⊗ $(ABC)^{-1} = (C^{-1}.B^{-1})A^{-1}$

A, B, C are nonsing.

A, B, C are nonsing.

* If $\underset{n\times n}{A}$ is nonsingular, then

A^T is nonsingular and $(A^T)^{-1} = \boxed{(A^{-1})^T}$

Proof: Assume A is nonsingular (A^{-1} exists)

$$A^T (A^{-1})^T = A^T (A^{-1})^T$$
$$= (A^{-1} A)^T$$

$\boxed{(CD)^T = D^T C^T}$

also $(A^{-1})^T A^T = I$. $= I^T = I$ $\quad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

so A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

1.5 Elementary matrices

* I nonsingular since $I\,I = I$

$I^{-1} = I$

Ex

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \boxed{E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

elementary matrix of type I

* An elementary matrix is a matrix $\underset{n\times n}{E}$ that is obtained from I_n by applying one E.R. operation

once. $\Longrightarrow$ 3 Types

$I_n \xrightarrow{R_i \leftrightarrow R_j} E$ elem. of type I

$I_n \xrightarrow{\alpha R_i} E$ " " " II

$I_n \xrightarrow{cR_i + R_j} E$ " " " III

Ex: $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \xleftarrow{-2R_2+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_n$

is elementary of type III

$\boxed{E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}} \longleftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

not elementary

Ex. $\boxed{E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \longleftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ elementary of type II

* Type I

$E = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & 1 & \\ 0 & & 1 & & \\ \vdots & 1 & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$ $\boxed{n \times n}$ $\longleftarrow \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots 0 \\ 0 & 0 & \cdots & 1 \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$

Type II

$E = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \alpha \\ & & & & 1 \end{pmatrix}$

Type $\quad E = \begin{pmatrix} 1 & & & \\ & \alpha & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}_{n \times n}$

Type III $\quad E = \begin{pmatrix} 1 & & & 0 \\ 0 & \ddots & & \\ \vdots & \boxed{c} & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

Ex: $\boxed{E = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$ ← elem. of type III $\quad I$

$\boxed{(2R_3 + R_1) \longleftrightarrow R_1}$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+2a_{31} & a_{12}+2a_{32} & a_{13}+2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$A \updownarrow 2R_3 + R_1$

$\begin{pmatrix} a_{11}+2a_{31} & & \\ & & \\ & & \end{pmatrix}$

* $I \longrightarrow IA = A$

$\downarrow$

$E \longrightarrow EA \neq A$

* If E is elementary matrix, then $\boxed{EA}$ (left multiplication by A) has the same result as applying the elementary row op. of E on A.

* $AE \longrightarrow$ column operation

* I nonsingular.

  E elem $\longrightarrow$ Is E nonsingular

  if nonsing, is E^{-1} elem.?

* Th. If E is elementary, then E is nonsingular and E^{-1} is elementary of the same type as E.

* E of type I

  $R_i \longleftrightarrow R_j$

$$E = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}$$

$E E = I$

$\Longrightarrow E$ is nonsingular and E^{-1} is elem. of same elem. $\Rightarrow E$ is nonsingular and $E^{-1} = E$ is elem. of same type.

Theorem. (Equivalent Conditions for Nonsingularity)

Let A be $n \times n$ matrix. The following are equivalent:

(a) A is nonsingular.

(b) $\boxed{Ax=0}$ has only the trivial solution 0.

(c) A is row equivalent to I.

Corollary. The $n \times n$ system $\boxed{Ax=b}$ has a unique solution $\Leftrightarrow$ A is nonsingular.

$Ax=b$, A nonsingular (A^{-1} exists)

has unique sol.

$$A^{-1}(Ax=b) \Rightarrow \boxed{x=A^{-1}b} \text{ sol.}$$

Ex:

$$\boxed{\begin{aligned} x_1 + x_3 &= 1 \\ -x_1 + x_2 + x_3 &= 2 \\ -2x_2 - 3x_3 &= 0 \end{aligned}}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -3 \end{pmatrix}$$ Nonsingular

$$\boxed{A^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ -3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix}}$$

solution $x = A^{-1} b = \begin{pmatrix} -1 & -2 & -1 \\ -3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -5 \\ -9 \\ 6 \end{pmatrix}}$

How to find A^{-1} (if exists)

Remark. If $\boxed{A \text{ is nonsingular}}$ then A is row equivalent to I. Hence, there exist elementary matrices $E_1, \dots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I \quad (1) \quad A^{-1}$$

Multiply both sides of this eq. by A^{-1}

$$\boxed{E_k E_{k-1} \cdots E_1 I = A^{-1}} \quad (2)$$

$$(A \mid I) \xrightarrow{E.R.O} \xrightarrow{A \text{ nonsing}} (I \mid A^{-1})$$

$$\xrightarrow{A \text{ sing}} (\neq I \mid X)$$

Ex: $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -3 \end{pmatrix}$. Is A nonsingular, if yes find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & \boxed{1} & 2 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & -3 & -3 & -2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right)$$

$I =$

$\therefore$ A is nonsingular and $A^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ -3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, Is A nonsingular, if yes find A^{-1}

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 3 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

$\neq I$

So $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ $\boxed{\text{is singular.}}$

$\neq I$ ... is singular.

* **LU-factorization:**

$$L=\begin{pmatrix}1&0&0\\-1&1&0\\3&2&5\end{pmatrix} \qquad U=\begin{pmatrix}2&1&-1\\0&-1&0\\0&0&3\end{pmatrix}$$

lower triang. upp. triang.

* Given $A_{n\times n}$, can we write $\boxed{A=LU}$

* Yes! If A can be transformed to an upper triangular matrix using row operation III only.

Ex: $A=\begin{pmatrix}2&4&2\\1&5&2\\4&-1&9\end{pmatrix}$, find LU-factoriz. of A. $(A=LU)$

$$\xrightarrow[-2R_1+R_3]{-\frac{1}{2}R_1+R_2}\begin{pmatrix}2&4&2\\ \boxed{0}&\boxed{3}&1\\ \boxed{0}&-9&5\end{pmatrix}\xrightarrow{3R_2+R_3}\boxed{\begin{pmatrix}2&4&2\\0&3&1\\0&\boxed{0}&8\end{pmatrix}}$$

so $\boxed{A=LU}=\begin{pmatrix}1&0&0\\1/2&1&0\\ \boxed{2}&-3&1\end{pmatrix}\begin{pmatrix}2&4&2\\0&3&1\\0&0&8\end{pmatrix}$

$l_{21}=1/2$, $l_{31}=2$, $l_{32}=-3$

Ex: $A=\begin{pmatrix}0&1&2\\1&-1&3\\2&1&1\end{pmatrix}$, LU-factorization

$$\begin{pmatrix}0&1&2\\1&-1&3\\2&1&1\end{pmatrix}\longrightarrow\begin{pmatrix}&&\\&&\\&&\end{pmatrix}$$

No

Ex: $A=\begin{pmatrix}1&-1&2\\1&2&1\\0&0&2\end{pmatrix}$, find LU-factor.

$$\xrightarrow{-R_1+R_2}\begin{pmatrix}1&-1&2\\ \dots&\dots&\dots\end{pmatrix}$$ so $A=\begin{pmatrix}1&0&0\\1&1&0\\ \dots\end{pmatrix}\begin{pmatrix}1&-1&2\\0&4&-1\\ \dots\end{pmatrix}$

$$\xrightarrow[0R_1+R_2]{-R_1+R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & \boxed{4} & -1 \\ 0 & \boxed{0} & 2 \end{pmatrix} \quad \text{so } A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}}_{U}$$

Chapter 2 Determinant

of square matrices: $A_{n\times n}$

Is A nonsingular? $(A|I) \longrightarrow (\overset{?}{I}| \;)$

Def: of $\det(A_{n\times n}) = |A|$

1) $A_{1\times1} = (a_{11}) \longrightarrow \det(A) = a_{11}$

2) $A_{2\times2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \det(A) = \boxed{a_{11}a_{22} - a_{12}a_{21}}$

red. $\downarrow$ (I) ; A is nonsingular $\Longleftrightarrow \det(A) \neq 0$

3) $A_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

A nonsingular $\Longleftrightarrow \det(A) \neq 0$.

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* cofactor $:= \boxed{A_{ij}} = (-1)^{i+j}\, M_{ij}$

where M_{ij} is the matrix obtained from A by deleting the ith row and jth column.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}_{3\times 3}$, $A_{13} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = -3$

$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -(-1) = 1.$

$A_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 3$, $A_{12} = (-1)^3 \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = -3$

* $\det(A_{3\times3})$ $\quad A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det(A_{3\times3}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$\det(A) \overset{*}{=}$ {columns

$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ $\quad = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ $\quad = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

$\boxed{A_{n\times n}}$ we define $\det(A)$ as

$\boxed{\det(A)} = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$ (ith row)

$= a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$ (jth column)

* A is nonsingular $\iff \det(A) \neq 0$

A is singular $\iff \det(A) = 0$

* A is nonsingular $\iff \det|A| \neq 0$
* A is singular $\iff \det|A| = 0$.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, find $\det(A)$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 1\cdot(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 - 2(-6) + 3(-3) = 0$$

$\therefore$ A is singular.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, find $\det(A)$.

$$\det(A) = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = 0 + 0 + 6(1)\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 24 \neq 0$$

$\therefore$ A is nonsingular. $\boxed{A^{-1}?}$

* If $\underset{n\times n}{A}$ is triangular (upper or lower), then $\det(A)$ = product of elements on main diagonal of A.

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \\ \end{pmatrix}$, $\det(A) = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & & \end{vmatrix}$

$$A = \begin{pmatrix} a_{11} & & & \\ \vdots & a_{22} & & \\ & & \ddots & \\ 0 & 0 \cdots 0 & & a_{nn} \end{pmatrix}, \quad \det(A) = a_{11} \begin{vmatrix} a_{22} & \cdots & \\ \vdots & \ddots & \\ 0 & \cdots 0 & a_{nn} \end{vmatrix}$$

$$= a_{11}\left(a_{22}\right) \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ 0 & & \\ \vdots & \ddots & \\ 0 & \cdots 0 & a_{nn} \end{vmatrix}$$

$$\vdots$$

$$= a_{11} a_{22} \cdots a_{nn}$$

* If $A_{n\times n}$ has a row (or column) of zeros,

$$\det(A) = 0$$

* If $A_{n\times n}$ has two identical rows (or two identical columns), then $\det(A) = 0$.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 3 & 5 & 6 & 7 \end{pmatrix}$, $\det(A) = 0$

singular.

* $\det(A_{n\times n}) = \det(A^T)$

2.2

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 7 & -1 & 1 \end{pmatrix}$

Properties of det(A)

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

find $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 1(-1)\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$

$$a_{11}A_{31} \quad a_{12}A_{32} \quad a_{13}A_{33} = -\begin{vmatrix} 1 & 2 \\ \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 2(-1) - 7 + 9 = 0$$

Theorem: If A is $n\times n$-matrix, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & i=j \\ 0 & i\neq j \end{cases}$$

Effect of row operations on det(A)

* $\underset{3\times 3}{A} \longrightarrow$ 3 det of size $\boxed{2\times 2}$.
* $\underset{4\times 4}{A} \longrightarrow$ 4 dets of size $3\times 3 \longrightarrow$ 12 dets of size 2×2.
* $\underset{5\times 5}{A} \longrightarrow$ 5 dets of size $4\times 4 \longrightarrow$ 60 dets of size 2×2

$$|A| = \begin{vmatrix} 2 & 3 & -1 & 1 \\ -2 & 4 & 2 & 6 \\ 1 & -1 & 3 & 5 \\ 6 & -1 & 1 & 2 \end{vmatrix} = 2\underset{3\times 3}{\begin{vmatrix} \; \end{vmatrix}} - 3\underset{3\times 3}{\begin{vmatrix} \; \end{vmatrix}} + (-1)\underset{3\times 3}{\begin{vmatrix} \; \end{vmatrix}} - \underset{3\times 3}{\begin{vmatrix} \; \end{vmatrix}}$$

Summary of 2.2

Thursday, April 2, 2020 12:18 PM

Summary. If E is an elementary matrix, then

$$\det(EA) = \det E \det A.$$

$$\det E = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Effect of row operations:

I. interchanging two rows changes the sign of $\det A$.

II. multiplying a row by α, the det is multiplied by α

III. Adding a multiple of one row to another has no effect on $\det A$.

$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} \diagdown & & \\ & \diagdown & \\ & & \diagdown \end{pmatrix}$ upper tria

* If $A_{n\times n}$, $\xrightarrow[\text{operation}]{\text{row}} \boxed{EA}$, E is elem.

① Row operation I (E is of type I) ($R_i \leftrightarrow R_j$)

$\det(A) \rightsquigarrow \det(EA)$

$$\boxed{\begin{aligned} &* \ \det(E) = -1, \ E \text{ of type I.} \\ &* \ \det(EA) = -\det(A). \\ &* \ \det(EA) = \det(E)\det(A) \end{aligned}}$$

Ex: $\begin{vmatrix} 2 & -1 & 1 \\ -3 & 4 & -2 \\ 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ -3 & 4 & -2 \\ 2 & -1 & 1 \end{vmatrix}$

② Row operation II.

E elem. of type II (αR_i)

$$\begin{cases} \det(EA) = \alpha \det(A). \\ \det(E) = \alpha \\ \det(EA) = \det(E)\det(A). \end{cases}$$

Ex:

$$2\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & 8 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 8 & 12 & 16 \\ -1 & 2 & 3 \end{vmatrix}$$

* Row operation III $(cR_i + R_j)$

E of type III

$$\det(E) = 1$$

$$\det(EA) = \det(A)$$

$$\det(EA) = \det(E)\det(A)$$

examples

Thursday, April 2, 2020 1:20 PM

Ex: find
$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ -1 & 1 & 2 & -1 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 3 & -1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -2 & 4 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 4 \end{vmatrix}$$

$$= 1 \cdot 3 \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} = 3(14) = 42$$

Ex
$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} \quad -2R_1 + R_2$$

$$= - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 10 & 4 \end{vmatrix} = -(1)(0) = 0$$

$\therefore \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{pmatrix}$ is singular.

Ex:
$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 4 & 6 & 8 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = 0$$

Ex: $\underset{4\times 4}{A}$, $\det(rA) = r^4 \det(A)$

Ex. 4×4 | det(rA) = r^n det(A)

* $\det(rA_{n\times n}) = \boxed{r^n} \det(A)$.

* $\boxed{\text{If } A, B \text{ } n\times n\text{-matrices, then } \det(AB) = \det(A)\det(B)}$

* Ex: Show that If A is nonsingular, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof: Assume A is nonsingular $(\det(A) \neq 0)$

$$AA^{-1} = I \qquad I = \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$