

1.4 nonsingular matrices

A $n \times n$ nonsingular $\Leftrightarrow \exists B$ s.t. $n \times n$
 $AB = \underline{I = BA}$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{I}}$

~~$BA = I$~~ $\Rightarrow A$ is nonsingular and

$A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix},$ and B is nonsing.
 $B^{-1} = A.$

* If $\frac{C}{n \times n} \frac{D}{n \times n} = I = DC \Rightarrow$

- ① C is nonsing, $C^{-1} = D$ and
- ② D " " , $\underline{D^{-1} = C.}$

A nonsingular
 A^{-1} is nonsingular and
 $(A^{-1})^{-1} = A.$

* If A $n \times n$ has no inverse, we say

A is singular. | $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ singular

* Th If A, B are nonsingular, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: A, B nonsing. A^{-1} exists, B^{-1} exists

$$AA^{-1} = I = A^{-1}A, \quad BB^{-1} = I = B^{-1}B.$$

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AI A^{-1} = AA^{-1} = I_n$$

and $(B^{-1}A^{-1})(AB) = I$

so AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

* If A_1, A_2, \dots, A_k are nonsingular $n \times n$ -mat

then $A_1 A_2 \dots A_k$ is nonsingular and

$$(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \dots A_2^{-1} A_1^{-1}$$

$$\textcircled{\otimes} (ABC)^{-1} = (C^{-1} B^{-1}) A^{-1}$$

A, B, C are nonsing.

A, B, C are nonsing.

* If A is nonsingular, then A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$

Proof: Assume A is nonsingular (A^{-1} exists)

$$(A^T)(A^{-1})^T = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(CD)^T = D^T C^T$$

also $(A^{-1})^T A^T = I = I^T = I$

so A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

1.5 Elementary matrices

* I nonsingular since $I I = I$

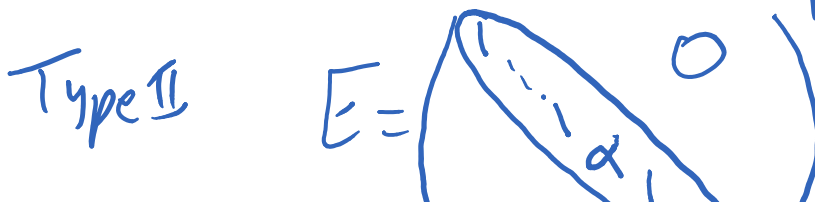
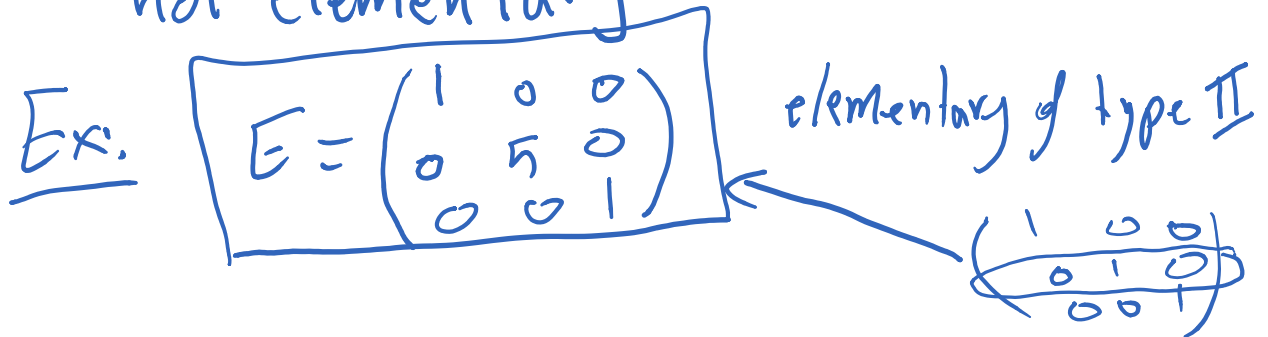
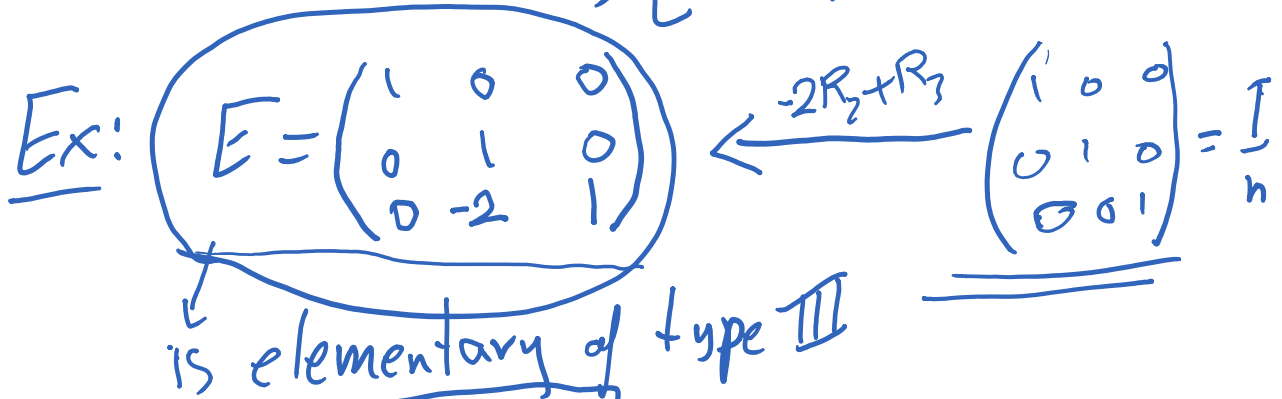
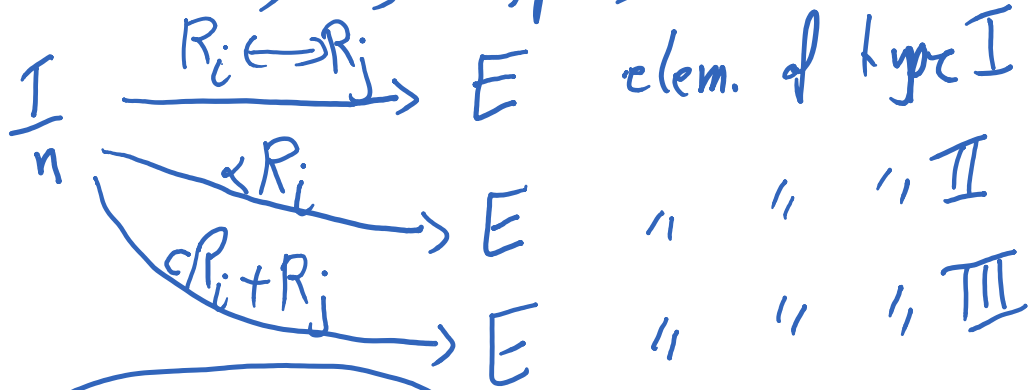
Ex $I_3^{-1} = I_3$

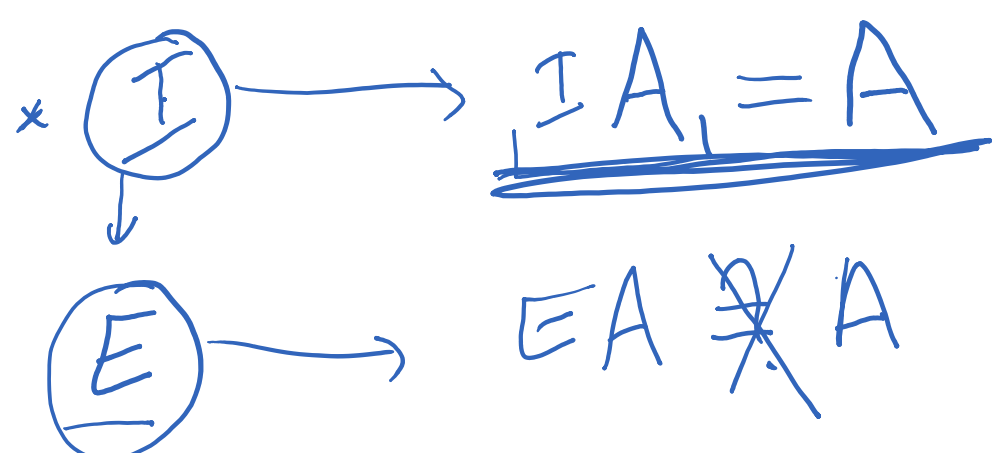
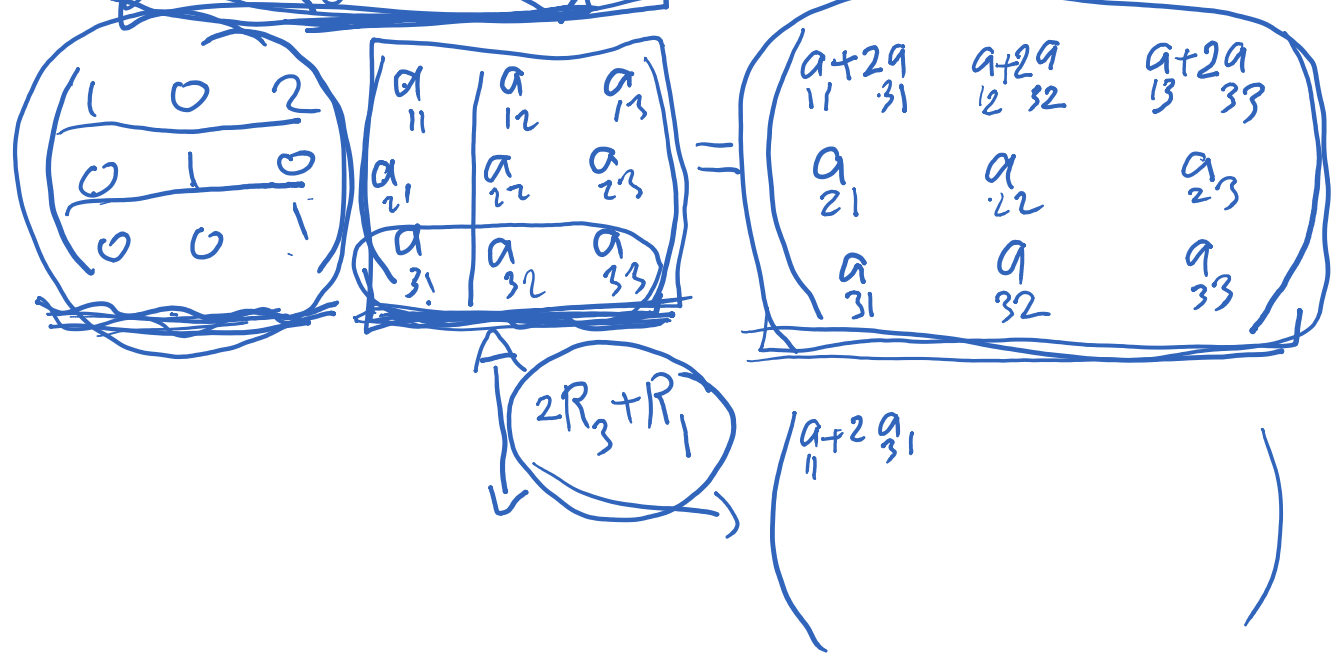
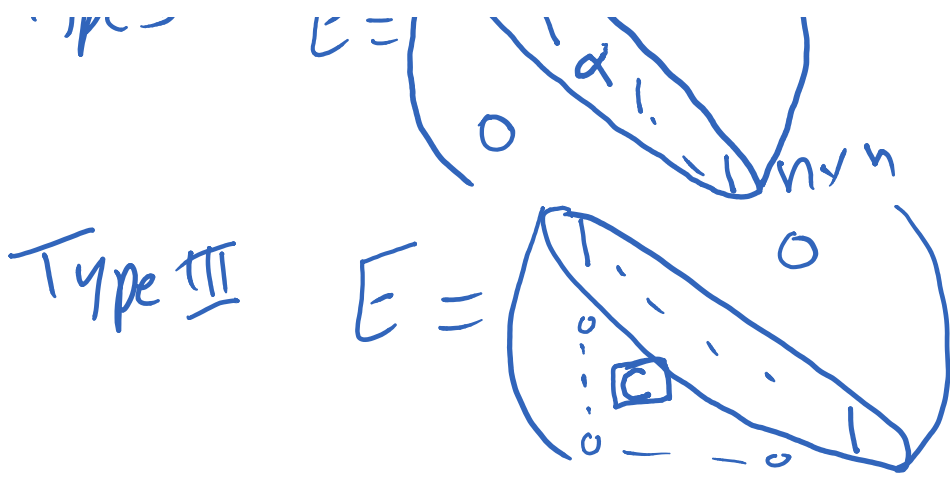
$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

elementary matrix of type I

* An elementary matrix is a matrix E that is obtained from I_n by applying E.R. operation one

once. \Rightarrow 3 Types





$(E) \rightarrow \dots$

* If (E) is elementary matrix, then (EA) (left multiplication by A) has the same result as applying the elementary row op. of E on A .

* $(AE) \rightarrow$ column operation

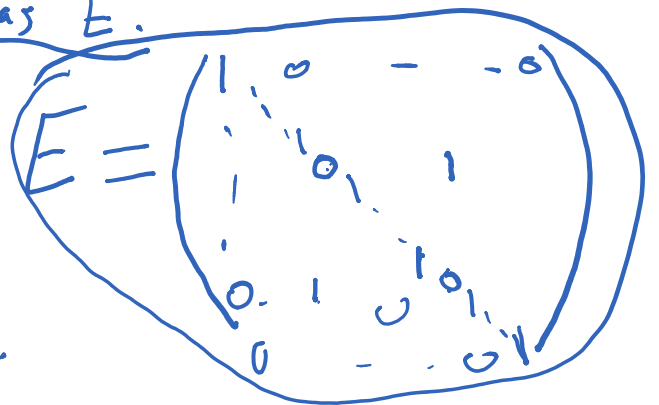
* I nonsingular.

$(E \text{ elem}) \rightarrow$ Is E nonsingular
if nonsing, is E^{-1} elem.?


* Th. If E is elementary, then E is nonsingular and (E^{-1}) is elementary of the same type as E .

* E of type I

$(R_i \leftrightarrow R_j)$



$(E^{-1}) = E$ is nonsingular and E^{-1} is elem of same

elem.  $\Rightarrow E$ is nonsingular and
 $E^{-1} = E$ is elem. of same type.

Theorem. (Equivalent Conditions for Nonsingularity)

Let A be $n \times n$ matrix. The following are equivalent!

- (a) A is nonsingular.
- (b) $Ax=0$ has only the trivial solution 0 .
- (c) A is row equivalent to I .

Corollary. The $n \times n$ system $Ax=b$ has a unique solution \Leftrightarrow A is nonsingular.

$Ax=b$, A nonsingular (A^{-1} exists)
 \rightarrow has unique sol.
 $A^{-1}(Ax=b) \Rightarrow$ $x = A^{-1}b$ sol.

Ex:

$$\begin{cases} x_1 + x_3 = 1 \\ -x_1 + x_2 + x_3 = 2 \\ -2x_2 - 3x_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -3 \end{pmatrix}$$

nonsingular

$$A^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ -3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

↓
solution $x = A^{-1}b = \begin{pmatrix} -1 & -2 & -1 \\ -3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -9 \\ 6 \end{pmatrix}$

How to find A^{-1} (if exists)

Remark. If A is nonsingular then A is row equivalent to I . Hence, there exist elementary matrices

E_1, \dots, E_k such that

$$(E_k E_{k-1} \dots E_1 A = I) \quad (1) \quad \bar{A}^{-1}$$

Multiply both sides of this eq. by A^{-1}

$$E_k E_{k-1} \dots E_1 I = A^{-1} \quad (2)$$

$$* \left(\begin{array}{c|c} A & I \end{array} \right) \xrightarrow{\text{E.R.O.}} \begin{array}{l} \xrightarrow{A \text{ nonsing}} \left(\begin{array}{c|c} I & \bar{A}^{-1} \end{array} \right) \\ \xrightarrow{A \text{ sing}} \left(\begin{array}{c|c} * & * \end{array} \right) \end{array}$$

Ex: $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -3 \end{pmatrix}$. Is A nonsingular, if

Yes find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 3 & -3 & -2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array} \right)$$

$\therefore A$ is nonsingular and $A^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ 3 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, Is A nonsingular, if yes

find A^{-1}

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 3 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 3 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & 1 \end{array} \right) \neq I$$

So $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ is singular.

~~is singular~~ $\neq I$ is singular.

* LU-factorization:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 5 \end{pmatrix}$$

lower triang.

$$U = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

upr. triang.

* Given $A_{n \times n}$, can we write $A=LU$

* Yes! If A can be transformed to an upper triangular matrix using row operation III only.

Ex! $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$, find LU-factoriz. of A . ($A=LU$)

$\begin{matrix} -R_1+R_2 \\ 2R_1+R_3 \end{matrix} \rightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{pmatrix} \xrightarrow{3R_2+R_3} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$

so $A=LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$

$l_{21} = \frac{1}{2}, l_{31} = 2, l_{32} = -3$

Ex! $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 1 & 1 \end{pmatrix}$, LU-factorization

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \quad \quad \quad \\ \quad \quad \quad \\ \quad \quad \quad \end{pmatrix}$ No

Ex! $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, find LU-factor.

$-R_1+R_2 \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ so $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

$\xrightarrow{-R_1+R_2}$
 $\xrightarrow{OR_1+R_2}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \text{ so } A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array} \right)$$

Chapter 2 | Determinant
of square matrices: $A_{n \times n}$
Is A nonsingular
 $(A|I) \rightarrow (I|\quad)$

Def: of $\det(A) = |A|$
 $n \times n$

1) $A = \begin{pmatrix} a_{11} \end{pmatrix}_{1 \times 1} \rightarrow \det(A) = a_{11}$

2) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2 \times 2} \rightarrow \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

red. $\left\{ \begin{array}{l} A \text{ is nonsingular} \Leftrightarrow \det(A) \neq 0 \end{array} \right.$

3) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3 \times 3}$

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A nonsingular $\Leftrightarrow \det(A) \neq 0$.

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* cofactor := $A_{ij} = (-1)^{i+j} |M_{ij}|$

where M_{ij} is the matrix obtained from A by deleting the i th row and j th column.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ $\rightarrow A = (-1)^{4|3} \begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = 3$

$A_{23} = (-1)^{5|2} \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -(-1) = 1$

$A_{11} = (-1)^{2|1} \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 3, \quad A_{12} = (-1)^{3|2} \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = -3$

* $\det(A)_{3 \times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det(A) = \int_{\text{columns}} \det(A)_{3 \times 3} = a_{11} \det(A_{11}) + a_{12} \det(A_{12}) + a_{13} \det(A_{13})$
 $= a_{11} \det(A_{11}) + a_{12} \det(A_{12}) + a_{13} \det(A_{13})$
 $= a_{12} \det(A_{12}) + a_{21} \det(A_{21}) + a_{31} \det(A_{31}) = a_{31} \det(A_{31}) + a_{32} \det(A_{32}) + a_{33} \det(A_{33})$

$A_{n \times n}$ we define $\det(A)$ as

$\det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$ (i-th row)
 $= a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$ (j-th column)

* A is nonsingular $\Leftrightarrow \det(A) \neq 0$

* A is nonsingular $\Leftrightarrow \det(A) \neq 0$
 * A is singular $\Leftrightarrow \det(A) = 0$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, find $\det(A)$

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 3 - 2(-6) + 3(-3) = 0 \\ \therefore A &\text{ is singular.} \end{aligned}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 6 & 6 \end{pmatrix}$, find $\det(A)$.

$$\det(A) = \begin{pmatrix} 1 \\ 31 \\ 31 \end{pmatrix} A_{11} + \begin{pmatrix} 2 \\ 32 \\ 32 \end{pmatrix} A_{12} + \begin{pmatrix} 3 \\ 33 \\ 33 \end{pmatrix} A_{13} = 0 + 0 + 6 \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 24 \neq 0$$

$\therefore A$ is nonsingular. $\boxed{A^{-1}}$

* If A is triangular (upper or lower), then
 $\det(A) =$ product of elements on main diagonal
 of A .

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}, \det(A) = a_{11} a_{22} \dots a_{nn}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}, \det(A) = a_{11} \begin{vmatrix} a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{vmatrix} = a_{11} a_{22} \dots a_{nn}$$

* If A has a row (or column) of zeros, $\det(A) = 0$

* If A has two identical rows (or two identical columns), then $\det(A) = 0$.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 3 & 5 & 6 & 7 \end{pmatrix}$, $\det(A) = 0$ singular.

* $\det(A) = \det(A^T)$

2.2

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 3 & -1 & 1 \end{pmatrix}$

Properties of $\det(A)$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Find $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2 \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 1(-1) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$

$$\begin{aligned}
 & \dots \quad \underbrace{11} \quad \underbrace{31} \quad \underbrace{12} \quad \underbrace{32} \quad \underbrace{13} \quad \underbrace{33} \quad \dots \quad \dots \quad \dots \quad \dots \\
 & = 2(-1) - 7 + 9 = 0
 \end{aligned}$$

$$\begin{aligned}
 & + 3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \\
 & \quad \quad \quad \dots \quad \dots \quad \dots \quad \dots
 \end{aligned}$$

Theorem: If A is $n \times n$ -matrix, then

$$\underbrace{a}_{i_1} A_{\dots \underbrace{j_1}_{j_1} \dots} + \underbrace{a}_{i_2} A_{\dots \underbrace{j_2}_{j_2} \dots} + \dots + \underbrace{a}_{i_n} A_{\dots \underbrace{j_n}_{j_n} \dots} = \begin{cases} \det(A) & i=j \\ 0 & i \neq j \end{cases}$$

Effect of row operations on $\det(A)$

- * $A_{3 \times 3} \rightarrow \underline{\underline{3}}$ dets of size 2×2
- * $A_{4 \times 4} \rightarrow \underline{\underline{4}}$ dets of size 3×3 $\rightarrow \underline{\underline{12}}$ dets of size 2×2
- * $A_{5 \times 5} \rightarrow \underline{\underline{5}}$ dets of size 4×4 $\rightarrow \underline{\underline{60}}$ dets of size 2×2

$$|A| = \begin{vmatrix} 2 & 3 & -1 & 1 \\ -2 & 4 & 2 & 6 \\ 1 & -1 & 3 & 5 \\ 6 & -1 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & -1 & 1 \\ -1 & 2 & 6 \\ -1 & 3 & 5 \end{vmatrix} - 3 \begin{vmatrix} -2 & 2 & 6 \\ 1 & 3 & 5 \\ 6 & 1 & 2 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 4 & 6 \\ 1 & -1 & 5 \\ 6 & -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 4 & 2 \\ 1 & -1 & 3 \\ 6 & -1 & 1 \end{vmatrix}$$

3×3
 3×3
 3×3
 3×3

Summary of 2.2

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Summary. If E is an elementary matrix, then

$$\underline{\det(EA)} = \underline{\det E \det A}.$$

$$\underline{\det E} = \begin{cases} \textcircled{-1} & \text{if } E \text{ is of type I} \\ \textcircled{\alpha \neq 0} & \text{if } E \text{ is of type II} \\ \textcircled{1} & \text{if } E \text{ is of type III} \end{cases}$$

Effect of row operations:

- I. interchanging two rows changes the sign of $\det A$.
- II. multiplying a row by α , the det is multiplied by α .
- III. Adding a multiple of one row to another has no effect on $\det A$.



* If A ($n \times n$) $\xrightarrow{\text{row operation}}$ EA , E is elem. upper tri.

① Row operation I (E is of type I) ($R_i \leftrightarrow R_j$)
 $\det(A) \rightsquigarrow \det(EA)$

- * $\det(E) = -1$, E of type I.
- * $\det(EA) = -\det(A)$.
- * $\det(EA) = \det(E)\det(A)$

Ex: $\begin{vmatrix} 2 & -1 & 1 \\ -3 & 4 & -2 \\ 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ -3 & 4 & -2 \\ 2 & -1 & 1 \end{vmatrix}$

② Row operation II
 E elem. of type II ($\times R_i$)

$\det(EA) = \alpha \det(A)$
 $\det(E) = \alpha$
 $\det(EA) = \det(E)\det(A)$

$\begin{vmatrix} 2 & -1 & 1 \\ -3 & 4 & -2 \\ 1 & 2 & 3 \end{vmatrix}$ $\begin{vmatrix} 2 & -1 & 3 \\ -3 & 4 & -2 \\ 1 & 2 & 3 \end{vmatrix}$

Ex:

$$2 \begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & 8 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 8 & 12 & 16 \\ -1 & 2 & 3 \end{vmatrix}$$

* Row operation III ($cR_i + R_j$)

E of type III

$$\det(E) = 1$$

$$\det(EA) = \det(A)$$

$$\det(EA) = \det(E) \det(A)$$

examples

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Ex: find $\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 3 & -1 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 4 \end{vmatrix}$

$= 1 \cdot 3 \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} = 3(14) = 42$

Ex: $\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} \quad (-2R_1 + R_2)$

$= - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 10 & 4 \end{vmatrix} = - (1)(0) = 0$

$\therefore \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{pmatrix}$ is singular.

Ex: $0 \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 4 & 6 & 8 \\ 1 & -1 & 1 \\ 4 & 6 & 8 \end{vmatrix} = 0$

Ex: $A_{4 \times 4}, \det(rA) = r^4 \det(A)$

U1

4x4

$$\det(rA_{n \times n}) = r^n \det(A).$$

* If A, B $n \times n$ -matrices, then
$$\det(AB) = \det(A) \det(B)$$

* Ex: Show that if A is nonsingular, then
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof: Assume A is nonsingular ($\det(A) \neq 0$)

$$A A^{-1} = \underline{I} \quad I = \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$
$$\det(A A^{-1}) = \det(I) = 1$$
$$\det(A) \det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$