3.6

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15/1.3) $A_{m\times n}$, A^TA
AA^T

$(A^T)A$ defined
$n\times\boxed{m}$ $\boxed{m}\times n$

$A(A^T)$ defined
$m\times\boxed{n}$ $\boxed{n}\times m$

3.6| $R(A)$, $C(A)$

$A_{m\times n} \xrightarrow{\text{reduction}} U$ $\boxed{(R)REF}$

* nonzero row of U form a basis for $R(A)$ (and for $R(U)$: $R(U)=R(A)$)

$\text{rank}(A) = \dim(R(A)) = \#$ of nonzero rows in U
$= \#$ of leading one's.

* $C(A) \neq C(U)$:

Determine the columns of U that have leading one's $\Longrightarrow$ The corresponding columns of A form a basis for $C(A)$.

$\dim(C(A)) = \#$ of leading ones in U.

$\Longrightarrow \dim(C(A)) = \dim(R(A)) = \text{rank}(A)$.

* Remarks: $A_{m\times n}$.

$\boxed{\begin{array}{l}\text{rank}(A) \le m \\ \text{rank}(A) \le n\end{array}}$ $\left\{ = \dim(R(A)) \right.$ $\left\{ = \dim(C(A)) \right.$

$\Longrightarrow \text{rank}(A) \le \min\{m, n\}$.

* $A_{5\times 7} \Longrightarrow \text{rank}(A) \le 5$.

$A_{4\times 2} \Longrightarrow \text{rank}(A) \le 2$.

* If $A_{m\times n}$. Let rank $= r$

1) If $r < m \longrightarrow$ rows of A are L.D.
2) If $r = m \longrightarrow$ rows of A are L.I.
3) If $r < n \Longrightarrow$ columns of A are L.D.
4) If $r = n \Longrightarrow$ columns of A are L.I.

Null space $= N(A_{m\times n}) = \{x \in \mathbb{R}^n : Ax = 0\}$ (all solutions to $Ax=0$).

Def: $\text{nullity}(A) = \dim(N(A))$.
$= \#$ of free variables.

Ex: Find $N(A)$, $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}} = U$

Find a basis and dimension of $R(A)$, $C(A)$, $N(A)$.

① Basis for $R(A) = \{(1 \ \ -2 \ \ 3), (0 \ \ 1 \ \ 5)\}$

$\text{rank}(A) = \dim(R(A)) = 2$

② Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ -4 \end{pmatrix} \right\}$, $\dim(C(A)) = 2$.

③ $N(A)$: $x_3 = \alpha$ free , $x_2 = -5\alpha$
$x_1 = 2x_2 - 3x_3 = -10\alpha - 3\alpha$
$= -13\alpha$.

$\therefore N(A) = \left\{ \begin{pmatrix} -13\alpha \\ -5\alpha \end{pmatrix} : \alpha \text{ scalar} \right\}$

③ $N(A)$: $x_3=\alpha$ free, $x_2=-5\alpha$

$x_1=2x_2-3x_3=-10\alpha-3\alpha=-13\alpha$.

$$\therefore N(A)=\left\{\begin{pmatrix}-13\alpha\\-5\alpha\\\alpha\end{pmatrix}: \alpha \text{ scalar}\right\}$$
$$=\left\{\alpha\boxed{\begin{pmatrix}-13\\-5\\1\end{pmatrix}}: \alpha \text{ scalar}\right\}.$$

Basis for $N(A)=\left\{\begin{pmatrix}-13\\-5\\1\end{pmatrix}\right\}$

nullity $=\dim N(A)=1$.

* # of leading variables $=$ rank(A).

\# of free variables $=$ nullity(A).

\# of leading variables $+$ # of free variables $=$ # of variables

$$\boxed{\text{rank}(A)+\text{nullity}(A)=n}\ (\text{number of columns})$$

Ex: $A=\begin{pmatrix}1&2&-1&-2\\-1&1&3&4\\-3&8&4&2\end{pmatrix}$. Find a basis and dimension of $R(A)$, $C(A)$, $N(A)$.

$$\longrightarrow\begin{pmatrix}1&2&-1&-2\\0&7&0&-2\\0&14&1&-4\end{pmatrix}\longrightarrow\begin{pmatrix}1&2&-1&-2\\0&7&0&-2\\0&0&1&0\end{pmatrix}=U$$

1) Basis for $R(A)=\{(1\ 2\ -1\ -2),(0\ 7\ 0\ -2),(0\ 0\ 1\ 0)\}$

rank$(A)=3$.

2) Basis for $C(A)=\left\{\begin{pmatrix}1\\-3\\-3\end{pmatrix},\begin{pmatrix}2\\1\\8\end{pmatrix},\begin{pmatrix}-1\\3\\4\end{pmatrix}\right\}$.

$\dim(C(A))=3$.

3) nullity$(A)=\dim(N(A))=1$.

$x_4=\alpha$ free, $x_3=0$, $7x_2=2x_4\Rightarrow x_2=\frac{2}{7}\alpha$

$x_1=-2\left(\frac{2}{7}\alpha\right)+0+2\alpha=\frac{10}{7}\alpha$.

$$N(A)=\left\{\begin{pmatrix}10/7\,\alpha\\2/7\,\alpha\\0\\\alpha\end{pmatrix}: \alpha \text{ scalar}\right\}$$

basis for $N(A)=\left\{\begin{pmatrix}10/7\\2/7\\0\\1\end{pmatrix}\right\}$

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* $\underset{m\times n}{A}x=b$

$Ax=b$ is consistent iff b can be written as a l.c. of columns of A.

$\iff b\in C(A)$.

$$\boxed{\ast\ Ax=b \text{ is consistent} \iff b\in C(A)}$$

* $\boxed{\underset{m\times n}{A}x=0}$ has only the zero solution $\iff$ $\boxed{x_1a_1+x_2a_2+\cdots+x_na_n=0}$ has only the zero solution $\iff$ $\boxed{\text{columns of } A \text{ are L.I.}}$

* $\underset{m\times n}{A}x=0$ has only the zero solution $\iff$ nullity$(A)=0\iff$ rank$(A)=n$

* $\underset{m\times n}{A}x=0$ has only the zero solution $\Longleftrightarrow$ nullity$(A)=0 \Longleftrightarrow$ rank$(A)=n$
$\Longleftrightarrow$ columns of A are L.I.

* $\underset{m\times n}{A}x=0$ has only the zero solution
$\Longleftrightarrow$ nullity $(A)=0$
$\Longleftrightarrow$ rank $(A)=n$
$\Longleftrightarrow$ columns of A are L.I.

* If $\underset{n\times n}{A}$: Then
$Ax=0$ has only the zero solution $\Longleftrightarrow$
$\Longleftrightarrow A$ $\boxed{\text{is nonsingular.}}$
$\Longleftrightarrow$ columns of A are $\boxed{\text{L.I.}}$ (n column $\in \mathbb{R}^n$)
$\Longleftrightarrow$ rank $(A)=n$
$\Longleftrightarrow$ nullity $(A)=0$.
$\Longleftrightarrow$ rows are L.I.
$\Longleftrightarrow$ columns of A form a basis for $\mathbb{R}^n$.
$\Longleftrightarrow$ rows of A form a basis for $\mathbb{R}^{1\times n}$.
$\Longleftrightarrow$ $|A|\neq 0$
$\Longleftrightarrow$ $Ax=b$ has a unique solution.

Th. Let A be $m\times n$-matrix
1) The system $Ax=b$ is consistent for every $b\in\mathbb{R}^m$ iff the columns of A form a spanning set of $\mathbb{R}^m$.
2) The system $\boxed{Ax=b}$ has at most one solution iff the columns of A are L.I.

Ex. 13 / 3.6) Let $\underset{4\times 3}{A}$ and $\left\{ z_1=\begin{pmatrix}1\\1\\2\end{pmatrix}, z_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$ basis for $N(A)$.
If $\boxed{b=a_1+2a_2+a_3}$. Find all solutions to $\boxed{Ax=b}$.

$N(A)=\left\{ \alpha\begin{pmatrix}1\\1\\2\end{pmatrix}+\beta\begin{pmatrix}1\\0\\-1\end{pmatrix} : \alpha,\beta \text{ scalars}\right\}$.

Since $b=a_1+2a_2+a_3 \Longrightarrow Ax=b$ is consistent.
and a solution to $Ax=b$ is $\boxed{x_0=\begin{pmatrix}1\\2\\1\end{pmatrix}}$

* If $Ax=b$ is consistent and x_0 is a particular sol
... solution to $Ax=b$ can be written as

then any solution to $Ax=b$ can be written as
$x = \boxed{x_0} + z$, z solution to $Ax=0$.

all solutions of $Ax=b$ have the form

$$x = x_0 + z = \boxed{\begin{pmatrix}1\\2\\1\end{pmatrix}} + \boxed{\alpha\begin{pmatrix}1\\2\\2\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-1\end{pmatrix}}$$

$$x = \left\{\begin{pmatrix}1+\alpha+\beta\\2+2\alpha\\1+2\alpha-\beta\end{pmatrix} : \alpha, \beta \text{ scalars}\right\}$$

all solutions to $Ax=b$

* 16) 3.6) $\boxed{\underset{5\times 8}{A}}$, $\boxed{\text{rank}(A)=5}$,
How many solutions does $\boxed{Ax=b}$ have?

$\Longrightarrow$ rows of A are L.I.

$\Longrightarrow$ nullity of $A = 3$.

* $\boxed{Ax=b}$ has infinite # of solutions (explain)

Ls consistent.

$(A \mid b) \longrightarrow$

no row of the form $(0\ 0 \cdots 0 \mid c)$, $c \neq 0$

so $Ax=b$ is consistent (and underdet.)

$\Longrightarrow$ $Ax=b$ has inf. # of sol.

* Ex: If $\underset{5\times 6}{A}$ and $Ax=b$ is consistent for all $b \in \mathbb{R}^5$

True or false:

- $\boxed{\text{nullity}(A)=0}$. $\longrightarrow$ False?
- $\text{rank}(A) = 1$. False.
- $\text{rank}(A) = 5 \longrightarrow$ True.
- $\text{nullity}(A) = 6. \longrightarrow$ False.

* $Ax=b$ is consistent $\Longleftrightarrow$ 6 columns of A form a sp. set for $\boxed{\mathbb{R}^5}$

\# of columns $=6 >$ $\dim(\mathbb{R}^5) = 5$ $\Longrightarrow$ 6 columns of A are L.D.

can be reduced to a basis.

# of columns $= 6$ $\Rightarrow$ $\dim(\mathbb{R}^5) = 5$ $\Rightarrow$ 6 columns of A are L.D.

$\Rightarrow$ can be reduced to a basis.

$\Rightarrow$ 5 columns are L.I.

$\Rightarrow$ rank$(A) = 5$ $\Rightarrow$ nullity$(A) = 1$.