14-5: 6.1 Eigenvalues and eigenvectors

Thursday, May 14, 2020 12:08 PM

Ex: $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$Ax = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3x.$$

$$Ax = 3x.$$

scalar other 3

$y = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\boxed{Ax = \lambda x}$$

$$Ay = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 3y$$

Def: Let A be $n\times n$-matrix. A scalar λ (complex) is called an eigenvalue for A if there exists a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$.

This $x \neq 0 \in \mathbb{R}^n$ (if exists) is called eigenvector for λ.

* Given $A_{n\times n}$: λ is an eigenvalue for A

$\iff \exists\, x \neq 0 \in \mathbb{R}^n$ s.t $\boxed{Ax = \lambda x}$

$\iff \exists\, x \neq 0 \in \mathbb{R}^n$ s.t $\boxed{(A-\lambda I)x = 0}$

$\iff \exists$ a nonzero solution to the homog. system $(A-\lambda I)x = 0$.

$\iff (A-\lambda I)$ is singular.

$\iff \boxed{\det(A-\lambda I) = 0.}$

Def: $\det(A-\lambda I) = p(\lambda)$ characterstic polynomial of A.

(polynomial of degree n).

* roots of $p(\lambda)$ {solution to $p(\lambda)=0$} are the eigenvalues of A.

* roots of $p=0$
of A.

Ex: $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$. Find eigenvalues of A and the corresponding eigenvectors

$\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix}$

$= (4-\lambda)(1-\lambda) + 2 = 4 - 5\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6.$

* solve $p(\lambda) = 0 \iff \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0.$

$\Rightarrow \lambda_1 = 2, \ \lambda_2 = 3$ eigenvalues of A.

① For $\boxed{\lambda_1 = 2}$ (Eigenvectors)

{Nonzero solutions of $(A - \lambda I)x = 0$ are eigenvectors}

solve $(A - 2I)x = 0$

$A - 2I = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

singular

$x_2 = \alpha$ free

$x_1 = x_2 = \alpha.$

solutions $x = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, α scalar.

Eigenvectors are $\boxed{x = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \ \alpha \neq 0.}$

* all solutions to $(A - \lambda I)x = 0$ are $N(A - \lambda I)$ {Null space} is called eigenspace for λ. $\{E(\lambda)\}$

$E(\lambda_1 = 2) = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalar} \right\} = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \alpha \text{ scalar} \right\}$

basis for $E(\lambda_1 = 2)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $\dim(E(\lambda_1 = 2)) = 1.$

* For $\lambda_2 = 3$

$\left\{ A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \right\}$

$A - 3I = \begin{pmatrix} 4-3 & -2 \\ 1 & 1-3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$x_2 = \beta$ free,

$x_1 = 2\beta$

$E(\lambda_2 = 3) = \left\{ \begin{pmatrix} 2\beta \\ \beta \end{pmatrix} : \beta \text{ scalar} \right\}.$

$= \left\{ \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix} : \beta \text{ scalar} \right\}$

basis for Eigenspace of $\lambda_2 = 3$ is $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, $\dim(E(\lambda_2 = 3)) = 1.$

basis for Eigenspace of $\lambda_2=3$ is $\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}$, $\dim(E(\lambda_2))=1$.

Eigenvectors $\begin{pmatrix}3\beta\\ \beta\end{pmatrix}$, $\beta\neq 0$.

* **Remark:** * $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn}\end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22}-\lambda & a_{23}\ \cdots & a_{2n}\\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots\ a_{n(n-1)} & a_{nn}-\lambda \end{vmatrix} = p(\lambda)\ :\ \text{of degree } n. \text{ polynomial.}$$

* roots of $p(\lambda)$ {solutions to $p(\lambda)=0$} are n roots (counting multiplicity) some roots are repeated.

$\Rightarrow$ A has n eigenvalues (some are repeated)

* distinct eigenvalues (without multiplicity).

Ex: $A=\begin{pmatrix}2 & -3 & 1\\ 1 & -2 & 1\\ 1 & -3 & 2\end{pmatrix}$. $p(\lambda)=|A-\lambda I| = \begin{vmatrix}2-\lambda & -3 & 1\\ 1 & -2-\lambda & 1\\ 1 & -3 & 2-\lambda\end{vmatrix}$

$$= \begin{vmatrix}2-\lambda & -3 & 1\\ 1 & -2-\lambda & 1\\ 0 & -1+\lambda & 1-\lambda\end{vmatrix} = (2-\lambda)\left[(-2-\lambda)(1-\lambda)-(\lambda-1)\right]$$
$$-1\left[-3(1-\lambda)-(\lambda-1)\right]$$

$$=(2-\lambda)\left[-2+2\lambda-\lambda+\lambda^2-\lambda+1\right]-\left[-3+3\lambda-\lambda+1\right]$$
$$=(2-\lambda)(\lambda^2-1)-\left[-2+2\lambda\right] = 2\lambda^2-2-\lambda^3+\lambda+2-2\lambda$$
$$=-\lambda^3+2\lambda^2-\lambda$$
$$=\lambda(-\lambda^2+2\lambda-1)$$
$$=-\lambda(\lambda^2-2\lambda+1)$$
$$=-\lambda(\lambda-1)^2.$$

$p(\lambda)=-\lambda(\lambda-1)^2$

solve $p(\lambda)=0$ $=-\lambda\boxed{(\lambda-1)^2}$

$\boxed{\lambda_1=0,\quad \lambda_2=\lambda_3=1.}$

Eigenvectors (1) For $\lambda_1=0$ $\rightarrow$ $A-0I=\begin{pmatrix}2 & -3 & 1\\ 1 & -2 & 1\end{pmatrix}$

* Eigenvectors: ① For $\lambda_1 = 0 \longrightarrow A - 0I = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $x_3 = \alpha$ free, $x_2 = \alpha$, $x_1 = \alpha$

$E(\lambda_1 = 0) = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalar} \right\} = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \alpha \text{ scalar} \right\}.$

basis for $E(\lambda_1 = 0)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, $\dim(E(\lambda_1 = 0)) = 1.$

② For $\lambda_2 = \lambda_3 = 1$

$A - I = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_2 = \beta$, $x_3 = \gamma$ free

$x_1 = 3\beta - \gamma$.

$E(\lambda_2 = \lambda_3 = 1) = \left\{ \begin{pmatrix} 3\beta - \gamma \\ \beta \\ \gamma \end{pmatrix} : \beta, \gamma \text{ scalars} \right\}.$

$= \left\{ \beta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : \beta, \gamma \text{ scalars} \right\}.$

basis for $E(\lambda_2 = \lambda_3 = 1)$ is $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim(E(\lambda_2 = \lambda_3 = 1)) = 2.$

* Remark: Given $A_{n \times n} \longrightarrow$ n eigenvalues $\boxed{\lambda_1, \lambda_2, \ldots, \lambda_n}$, (some are repeated)

$$\boxed{p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) = |A - \lambda I|}$$

* $p(0) = (\lambda_1)(\lambda_2) \cdots (\lambda_n) = |A - 0I| = |A|.$

$$\boxed{\text{product of all eigenvalues} = |A|}$$

* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace}(A) = \sum_{i=1}^{n} a_{ii}$ (sum of elements on main diagonal of A)

* $\lambda_1+\lambda_2+\cdots+\lambda_n = \text{trace}(A)$ $\quad i=1$

Ex. $A=\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $\lambda_1=2,\ \lambda_2=3$

$\lambda_1\lambda_2 = 6$, $|A| = \begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} = 6$

$\boxed{p(\lambda)=\lambda^2-5\lambda+6}$.

$\lambda_1+\lambda_2=5$

$\text{trace}(A)=5$

Ex. $\boxed{A=\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}}$ is singular.

$\boxed{\lambda_1=0}$, $\lambda_2=\lambda_3=1$.

1) $\lambda_1\lambda_2\lambda_3 = 0 = \det(A)$

2) $\lambda_1+\lambda_2+\lambda_3 = 0+1+1 = 2$.

$\text{trace}(A) = 2$

* A is singular $\Longleftrightarrow$ $\lambda=0$ is an eigenvalue of A.
* A is nonsingular $\Longleftrightarrow$ $\lambda=0$ is not an eigenvalue for A.

Ex: $A=\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ triangular.

$p(\lambda) = |A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(2-\lambda)$

$= (2-\lambda)^2(3-\lambda)$.

Eignvalues are $\lambda_1=\lambda_2=2$, $\lambda_3=3$.

* If A is triangular (upper or lower), then eigenvalues of A are the elements of main diagonal of A.

* Ex: $A=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, Find Eigenvalues and Eigenvectors.

$p(\lambda)=|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+4$

solve $p(\lambda)=0 \Rightarrow (1-\lambda)^2+4=0 \Rightarrow \lambda^2-2\lambda+5=0$.

$\lambda = \dfrac{2\pm\sqrt{4-20}}{2} = 1\pm 2i$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$\boxed{\lambda_1 = 1+2i,\ \lambda_2 = 1-2i}$ → Eigenvalues.

① $\lambda_1 = 1+2i$: $(A-(1+2i)I) = \begin{pmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{pmatrix}$

$$= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \xrightarrow{-iR_1+R_2} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

$x_2 = \alpha$ free

$x_1 = -i\alpha$

$$E(\lambda_1 = 1+2i) = \left\{ \begin{pmatrix} -i\alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalar} \right\},$$

basis for $E(\lambda_1 = 1+2i)$ is $\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$ $(\alpha = 1)$

or basis is $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$ $(\alpha = i)$

$\boxed{\begin{array}{l} z = \begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ e.v. } \lambda_1 \\ \bar{z} = \begin{pmatrix} i \\ 1 \end{pmatrix} \end{array}}$

② For $\lambda_2 = 1-2i$

$$A-(1-2i)I = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 1-2i & 0 \\ 0 & 1-2i \end{pmatrix} = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \xrightarrow{-iR_1+R_2} \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

$x_2 = \beta$ free.

$$ix_1 = -\beta \Longrightarrow x_1 = -\frac{\beta}{i}\left(\frac{-i}{-i}\right) = \frac{i\beta}{1}$$

$$E(\lambda_2 = 1-2i) = \left\{ \begin{pmatrix} i\beta \\ \beta \end{pmatrix} : \beta \text{ scalar} \right\}$$

basis for $E(\lambda_2 = 1-2i)$ is $\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$

or $\left\{ \begin{pmatrix} -1 \\ i \end{pmatrix} \right\}$

* $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ real matrix,

$\lambda_1 = 1+2i$, $\lambda_2 = 1-2i$ $\left(\underline{\lambda_2 = \bar{\lambda}_1}\right)$

$\text{Trace}(A) = \lambda_1 + \lambda_2 = 1 + \cancel{2i} + 1 - \cancel{2i} = 2$

$\det(A) = \lambda_1 \lambda_2 = (1-2i)(1+2i) = 1 + \cancel{2i} - \cancel{2i} - 4(-1) = 5$

$\det(A) = 5$

* If A is real matrix and $\lambda = a+ib$ is a complex eigenvalue of A, then $\bar{\lambda} = a - ib$ is also an eigenvalue of A.

* If z is eigenvector for $\lambda = a+ib$, then $\bar{z}$ is eigenvector for $\bar{\lambda}$.

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* If λ is eigenvalue for A, with eigenvector $x \neq 0$

$$\Longrightarrow \quad \underline{Ax = \lambda x}$$

$$\Longrightarrow \boxed{A^2} x = A \lambda x = \lambda \underbrace{Ax} = \lambda(\lambda x) = \lambda^2 x$$

so λ^2 is eigenvalue for A^2 with same eigenvector x as λ.

* λ^k is eigenvalue for A^k with eigenvector $\underline{x}$.

Def: Similar matrices:

A matrix $A_{n \times n}$ is called similar to a matrix $B_{n \times n}$ if there exists a nonsingular matrix X such that

$A = X^{-1} B X$ ($\Longleftrightarrow$ $B = X A X^{-1}$)

$$A = X^{-1} B X . \quad (\Longleftrightarrow \quad B = X A X^{-1})$$

Th. If A, B are similar matrices, then they have the same characterstic polynomial and so same eigenvalues.

Proof: Assume A, B are similar

$$\Longrightarrow \quad A = X^{-1} B X.$$

$$P_A(\lambda) = |A - \lambda I| = |X^{-1} B X - \lambda I| \qquad I = X^{-1} X$$

$$= |X^{-1} (B - \lambda I) X|$$

$$= |X^{-1}| \, |B - \lambda I| \, |X| \qquad , |X| \neq 0$$

$$= \frac{1}{\cancel{|X|}} |B - \lambda I| \, \cancel{|X|}$$

$$= |B - \lambda I| = P_B(\lambda).$$