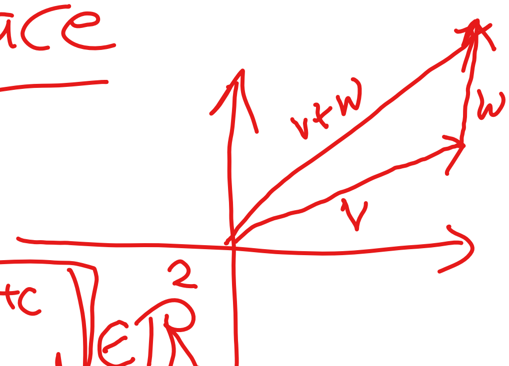
Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, we can associate a unique element $\boxed{\mathbf{x}+\mathbf{y}}$ that is also in V, and with each element $\mathbf{x}$ in V and each scalar α, we can associate a unique element $\boxed{\alpha\mathbf{x}}$ in V. The set V together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied:

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.
A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ in V. associative law
A3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x} \in V$.
A4. For each $\mathbf{x} \in V$, there exists an element $-\mathbf{x}$ in V such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V.
A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x} \in V$.
A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x} \in V$.
A8. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x} \in V$.

$\mathbb{R}^2$ $\begin{pmatrix}0\\0\end{pmatrix}$

Chapter 3: Vector Space



Ex: $\mathbb{R}^2=\left\{\begin{pmatrix}x\\y\end{pmatrix} x,y\in\mathbb{R}\right\}$

$\vec{v}=\begin{pmatrix}a\\b\end{pmatrix}, \vec{w}=\begin{pmatrix}c\\d\end{pmatrix}: \boxed{v+w}=\boxed{\begin{pmatrix}a+c\\b+d\end{pmatrix}}\in\mathbb{R}^2$

$\alpha\vec{v}=\alpha\boxed{\begin{pmatrix}a\\b\end{pmatrix}}=\begin{pmatrix}\alpha a\\\alpha b\end{pmatrix}\in\mathbb{R}^2$

$\boxed{\begin{array}{l} v+w=w+v \\ \alpha(v+w)=\alpha v+\alpha w \end{array}}$

Ex: $\mathbb{R}^3=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}: x,y,z\in\mathbb{R}\right\}\neq\emptyset$

$\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\begin{pmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{pmatrix}\in\mathbb{R}^3$

$\alpha\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\alpha a\\\alpha b\\\alpha c\end{pmatrix}\in\mathbb{R}^3$

$\left\{\begin{array}{l} v+w=w+v \\ - \\ - \\ - \\ - \end{array}\right.$

Ex: $\boxed{\mathbb{R}^{2\times 2}} = \{A_{2\times 2} = (a_{ij})_{2\times 2}\} \neq \phi$

$$\underline{A_{2\times 2} + B_{2\times 2}} = C_{2\times 2} \in \underline{\mathbb{R}^{2\times 2}}$$

$$\underline{\alpha A} = (\alpha a_{ij})_{2\times 2} \in \underline{\mathbb{R}^{2\times 2}}$$

$\left\{ \begin{array}{l} A+B = B+A \\ \alpha(A+B) = \alpha A + \alpha B \\ \equiv \end{array} \right.$

* Ex: polynomials

$(p+q)(x)$

αp

function

$(f+g)(x)$

αf

Ex: $V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$

under $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \boxed{\in \mathbb{R}^n}$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \boxed{\in \mathbb{R}^n}$$

Is a vector space

1) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ✓

2) ✓

3) $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$: $\boxed{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ✓

$x + 0 = x$

4) Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $-x = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix} \in \mathbb{R}^n$

$$x + (-x) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$\vdots$

8) $1 \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

so $\mathbb{R}^n$ with operations defined above is a vector space.

Ex: $V = \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

define $\begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} b \\ 1 \end{pmatrix} := \begin{pmatrix} a+b \\ 1 \end{pmatrix} \in V$ nonstandard.

$\alpha \odot \begin{pmatrix} a \\ 1 \end{pmatrix} := \begin{pmatrix} \alpha a \\ 1 \end{pmatrix} \in V$

$\cancel{\mathbb{R}^2 \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}}$

$\begin{pmatrix} a \\ 1 \end{pmatrix} + \begin{pmatrix} b \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ 2 \end{pmatrix} \notin V$

not defined

not vector space

1) $\begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} b \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ 1 \end{pmatrix} = \begin{pmatrix} b+a \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ 1 \end{pmatrix} \oplus \begin{pmatrix} a \\ 1 \end{pmatrix}$

2) $\left(\begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} b \\ 1 \end{pmatrix} \right) \oplus \begin{pmatrix} c \\ 1 \end{pmatrix} := \begin{pmatrix} a+b \\ 1 \end{pmatrix} \oplus \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} a+b+c \\ 1 \end{pmatrix}$

$\parallel$

$\begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \left(\begin{pmatrix} b \\ 1 \end{pmatrix} + \begin{pmatrix} c \\ 1 \end{pmatrix} \right) = \begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} b+c \\ 1 \end{pmatrix} = \begin{pmatrix} a+b+c \\ 1 \end{pmatrix}$ =

3) $0 \in V \longrightarrow \boxed{\bar{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \in V = \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix} \right\}$

$0 \oplus x = x$

$$\bar{0} \oplus x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} a \\ 1 \end{pmatrix} := \begin{pmatrix} a \\ 1 \end{pmatrix} = \underline{x}$$

4) Let $x = \begin{pmatrix} a \\ 1 \end{pmatrix} \in V$, find $\underline{-x} = \begin{pmatrix} -a \\ 1 \end{pmatrix} \underline{\in V}$

s.t. $\underline{x \oplus -x} = \begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} -a \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{0}$

5) ✓ $\alpha \odot \left(\begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \begin{pmatrix} b \\ 1 \end{pmatrix} \right) = \alpha \odot \begin{pmatrix} a+b \\ 1 \end{pmatrix} = \begin{pmatrix} \underline{\alpha(a+b)} \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} \alpha a + \alpha b \\ 1 \end{pmatrix}$$

$$\alpha \odot \begin{pmatrix} a \\ 1 \end{pmatrix} \oplus \alpha \odot \begin{pmatrix} b \\ 1 \end{pmatrix} = \begin{pmatrix} \underline{\alpha a} \\ 1 \end{pmatrix} \oplus \begin{pmatrix} \underline{\alpha b} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha a + \alpha b \\ 1 \end{pmatrix}$$

8) $1 \odot \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix}$ $\boxed{6, 7}$

$\therefore V = \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$ under the above defined operations $\{\oplus, \odot\}$ is a vector space.

Ex: ① $\mathbb{R}^n$

② $P_n = \{ p(x) : p(x)$ is a polynomial of degree less than $n\}$

let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$.

$q(x) = b_0 + b_1 x + \cdots \cdots + b_{n-1} x^{n-1}$

$(p+q)(x) = a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1}) x^{n-1} \in P_n$

$(\alpha p)(x) = \alpha a_0 + (\alpha a_1)x + \cdots + (\alpha a_{n-1}) x^{n-1} \in P_n$

① ———— ⑧, $\boxed{0 = 0 + 0x + \cdots + 0x^{n-1}}$

* Ex: $P_3 = \{ p(x) : \deg(p(x)) < 3 \}$

$p(x) = 2 - x - 2x^2$ $\quad (\deg(p) = 2 <)$

$q(x) = 4 + x + 2x^2$. $\quad (\deg(q) = 2 < 3)$

$(p+q)(x) = 6 \longrightarrow \deg(p+q) = 0 < 3$

$(2 - x - 2x^2) + (-2 + 2x + 2x^2)$
$= x$

Ex: $M_{m\times n}(\mathbb{R}) = \mathbb{R}^{m\times n}$: The set of all $m\times n$-matrices

$A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$

* $A + B = (a_{ij} + b_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

α $\alpha A = (\alpha a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

conditions $\longleftrightarrow$ prev. theorem

$A+B=B+A$

* $0 = 0_{m\times n} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$: $0+A=A$.

* $A \in \mathbb{R}^{m\times n}$, $-A = (-a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

$A+(-A) = 0_{m\times n}$

$0 \in \mathbb{R}$.

$0 \in \mathbb{R}^{m\times n}$

$0 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$

$0 \in P_n$:

$0 = 0 + 0x + \cdots + 0x^{n-1}$

polynomials

If $\boxed{ax^2+bx+c = \boxed{0}}$ find a, b, c

$(0x^2+0x+0)$

$a=0$
$b=0$
$c=0$.

solve $\boxed{ax^2+bx+c} = 0$, find x $\in \mathbb{R}$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$