14-4 meeting

Tuesday, April 14, 2020 12:19 PM

Vector space: $(\underline{\underline{V}}, \underline{\underline{+}}, \underline{\underline{\cdot}})$

$\mathbb{R}^n$ $(+,\cdot)$, P_n, $\mathbb{R}^{m\times n}$

Ex: $C[a,b]$ = the set of all continuous functions on $[a,b]$ is a vector space.

under Let $f(x), g(x) \in C[a,b]$,

$(f+g)(x) = f(x)+g(x) \in C[a,b]$

$(\alpha f)(x) = \alpha (f(x)) \in C[a,b]$.

* If $f(x)$, $g(x)$ cont. on $[a,b] \Rightarrow f+g$ is cont. on $[a,b]$

cond. (1) $f+g = g+f$

⋮

(3) $0(x) = 0$, $\forall x \in [a,b]$ $\in C[a,b]$ and sati. $(0+f)(x) = f(x)$.

(4) $f(x) \in C[a,b]$, $-f(x) \in C[a,b]$

$f(x) + (-f(x)) = \underline{0}$, $\forall x \in [a,b]$

⋮

(8)

Properties: Th. Let V is a vector space and $x \in V$, then

1) $0x = 0_V$

2) If $x, y \in V$, and $\boxed{x+y = 0_V}$, then $y = -x$

2) If $x, y \in V$, and $x + y = 0_V$, then $y = -x$

3) $(-1)x = -x$

Proof: ① let $x \in V$,

$$x = 1 \cdot x = (1+0)x \overset{(5)}{=} 1 \cdot x + 0 \cdot x = x + 0 \cdot x$$

We know:
$$0_V = x + (-x) = (x + 0 \cdot x) + (-x)$$
$$= (0 \cdot x + x) + (-x)$$
$$= 0 \cdot x + (x + (-x))$$
$$= \boxed{0 \cdot x} + 0_V$$
$$= 0 \cdot x \quad (3)$$

$$\boxed{0 \cdot x = 0_V}$$

(2) If $x + y = 0_V$, then $y = -x$

Assume $\boxed{x + y = 0_V}$

$$\Rightarrow \; -x = -x + \boxed{0_V} = -x + (x + y)$$
$$= (-x + x) + y$$
$$= 0_V + y = y$$

$$\therefore \boxed{y = -x}$$

(3) $\boxed{(-1)x = -x}$

$$\boxed{x + \boxed{(-1)x}} = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x$$
$$= 0 \cdot x = 0_V$$

by (2) $\Rightarrow \boxed{(-1)x = -x}$

(4) Let V vector space, $v \in V$, α scalar and

$\boxed{\alpha \cdot v = 0_V}$, then $\boxed{\boxed{\alpha = 0} \text{ or } v = 0}$

$\alpha, \beta \in \mathbb{R}$

$\alpha\beta = 0 \Rightarrow \alpha = 0$ or $\beta = 0$.

Proof: Assume $\boxed{\alpha \cdot v = 0_V}$ and assume $\boxed{\alpha \neq 0}$

$\boxed{(\text{show } v = 0)}$

Since $\alpha \neq 0 \Rightarrow \frac{1}{\alpha} \in \mathbb{R}$

Since $\alpha \cdot v = 0_V$, multiply by $\frac{1}{\alpha}$ $\Rightarrow$

$$\left(\frac{1}{\alpha}\right)(\alpha \cdot v) = \boxed{\frac{1}{\alpha} 0_V} \Rightarrow$$

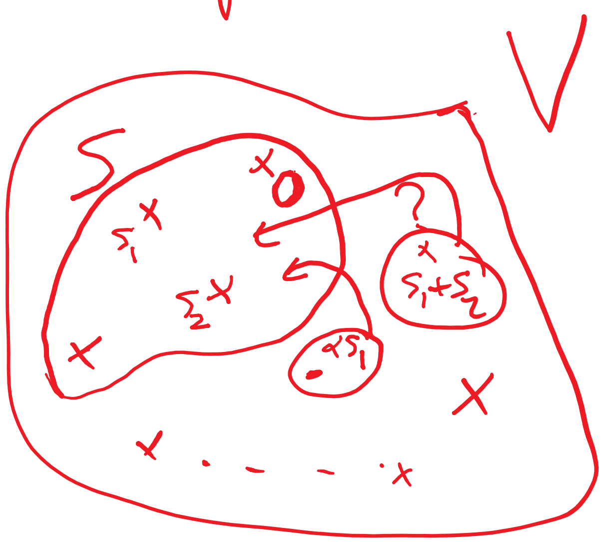
$$\boxed{\beta \cdot 0_V = 0_V}$$

$$\left(\alpha \left(\frac{1}{\alpha}\right)\right) v = 0_V \Rightarrow 1 \cdot v = 0_V$$

$$\Rightarrow \boxed{v = 0_V}$$

* $(V, +, \cdot)$ vector space

$\boxed{\begin{array}{c} s_1 + s_2 \overset{?}{\in} S \\ \alpha s_1 \overset{?}{\in} S \end{array}}$



3.2

Def: Let V be a vector space, $S \subseteq V$

$\boxed{S \neq \phi}$, we say S is a subspace,

1) for all $s_1, s_2 \in S$, we have $s_1 + s_2 \in S$.
2) for all $s \in S$, α scalar, we have $\alpha s \in S$.

Ex: Let $S = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$. Is S a subspace of $\boxed{\mathbb{R}^2}$

* $S \neq \phi$: $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$ ✓

1) Let $s_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \in S$

Now $s_1 + s_2 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \in S$ ✓

② Let $s_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in S$, α scalar.

$\alpha s_1 = \alpha \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix} \in S$ ✓

$\therefore$ S is a subspace of $\mathbb{R}^2$.

Ex: $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = x_1 + 1,\ x_1, x_2 \in \mathbb{R} \right\}$

Is S a subspace of $\mathbb{R}^2$.

$S = \left\{ \begin{pmatrix} x_1 \\ x_1 + 1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$. Is S a subspace of $\mathbb{R}^2$??

- $\boxed{S \neq \phi}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \overset{?}{\notin} S$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$

1) Let $s_1 = \begin{pmatrix} x_1 \\ x_1 + 1 \end{pmatrix}$, $s_2 = \begin{pmatrix} x_2 \\ x_2 + 1 \end{pmatrix} \in S$

$s_1 + s_2 = \begin{pmatrix} x_1 \\ \ldots \end{pmatrix} + \begin{pmatrix} x_2 \\ \ldots \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ \ldots \end{pmatrix} \overset{?}{\notin} S$

$$s_1+s_2=\begin{pmatrix}x_1\\x_1+1\end{pmatrix}+\begin{pmatrix}x_2\\x_2+1\end{pmatrix}=\begin{pmatrix}x_1+x_2\\x_1+x_2+2\end{pmatrix}\notin S$$

$\begin{pmatrix}1\\2\end{pmatrix}\in S$, $\begin{pmatrix}2\\3\end{pmatrix}\in S$, but $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}\notin S$

so S is not a subspace of $\mathbb{R}^2$

Ex: $S=\{f(x)\in C[0,1] : f(1)=0\}$.

Is S a subspace of $C[0,1]$.

- $S\neq\phi$ ✓ since $\boxed{0\in S}$ ✓ $(0(1)=0)$
- let $f(x), g(x)\in S$, so $f(1)=0$, $g(1)=0$.

Now $\boxed{f+g\overset{?}{\in} S}$: $(f+g)(1)=f(1)+g(1)=0+0=0$

$\therefore \boxed{f+g\in S}$

- Let $f(x)\in S$, α scalar, $f(1)=0$.

Now $(\alpha f)(1)=\alpha(f(1))=\alpha(0)=0$

$\therefore \boxed{\alpha f\in S.}$

so S is a subspace of $C[0,1]$.

Ex: $S=\{p(x)\in P_3 : p(0)=1\}$.

Is S a subspace of P_2.

- $S \neq \phi$ ✓ since $p(x) = x+1 \in S$ ✓
$\boxed{p(0)=1}$

- Let $p(x), q(x) \in S$, so $p(0)=1$, $q(0)=1$.

Now $(p+q)(0) = p(0) + q(0) = 1+1 = 2$

$\Rightarrow \boxed{p+q \notin S.}$

So S is not a subspace of P_3.

* Remark: If V is a vector space, S is a subspace of V, then $\boxed{0_V \in S.}$

Proof: S subspace $\Rightarrow S \neq \phi$

Let $s \in S$, $\alpha = 0 \in \mathbb{R}$.

$S \ni \alpha s = 0 s = 0_V$, $\Rightarrow \boxed{0_V \in S}$

* Let V be a vector space, $S \subseteq V$
If $0_V \notin S$, then S is not a subspace of V.

Ex: $S = \{p(x) \in P_3 : p(0)=1\}$. Is S a subspace of P_3?

- $S \neq \phi$.

$\boxed{0 \notin S}$

$\therefore$ S is not a subspace of P_3.

of I_3.

* Let $\underset{m\times n}{A}$ any matrix $\left\{ \boxed{\underset{m\times n}{A}x=0} \right.$

The set of all solution to $Ax=0$ is called the nullspace of A.

$$N(A)=\{x\in\mathbb{R}^n : Ax=0\}.$$

$$\boxed{N(A)=\left\{x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in\mathbb{R}^n : Ax=0\right\}}.$$

Thm $N(\underset{m\times n}{A})$ is a subspace of $\mathbb{R}^n$.

$$\boxed{N(A)=\left\{x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in\mathbb{R}^n : \boxed{Ax=0}\right\}}.$$

- $N(A)\neq\phi$ since $0=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\in N(A)$

(0 is always a sol. to $Ax=0$)

- let $y,z\in N(A) \Rightarrow Ay=0,\ Az=0$

$\boxed{y+z\overset{?}{\in}N(A)}$: consider $A(y+z)=Ay+Az$
$=0+0=0$

$\therefore\ y+z\in N(A).$

- Let $y \in N(A)$, α scalar $\Rightarrow Ay = 0$

$\boxed{\alpha y \overset{?}{\in} N(A)}$: $A(\alpha y) = \alpha(Ay) = \alpha 0 = 0$

$\therefore \underline{\alpha y \in N(A)}$.

so $N(A)$ is a subspace of $\mathbb{R}^n$.

Ex: If $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & -1 \\ 3 & -1 & 1 & 0 \end{pmatrix}$, Find $\underline{N(A)}$.

(solve $\boxed{Ax = 0}$)

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 1 & -1 & 0 \\ 3 & -1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 3 & -3 \\ 0 & -4 & 4 & -3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1 free variable: $x_3 = \alpha$

$x_4 = 0$, $x_2 = x_3 - x_4 = \alpha$

$x_1 = -x_2 + x_3 - x_4 = -\alpha + \alpha - 0 = 0$.

$$\therefore N(A) = \left\{ \begin{pmatrix} 0 \\ \alpha \\ \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \underline{\left\{ \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}}$$