3.2 subspaces

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$a_1, a_2, \dots, a_k$:

Def: Let $v_1, v_2, \dots, v_n \in V$, V a vector space.
A sum of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called a linear combination of $v_1, \dots, v_n$.

Ex: $v_1 = x^2 + x$, $v_2 = 2x+1$, $v_3 = x-1$ $\in P_3$.

A linear combination of v_1, v_2, v_3 is

$$3v_1 + 4v_2 - 2v_3 = 3(x^2+x) + 4(2x+1) - 2(x-1)$$
$$= 3x^2 + 9x + 6$$

$3x^2 + 9x + 6$ is a linear combination of v_1, v_2, v_3.

* Another one $0v_1 + 0v_2 + 0v_3 = 0$.
  $p(x) = 0$ is a l.c. of v_1, v_2, v_3.

* Is $\boxed{4x^2 - x + 2}$ a l.c. of v_1, v_2, v_3 ?

find $\boxed{\alpha_1, \alpha_2, \alpha_3}$ scalars s.t.

$$\boxed{4x^2 - x + 2} = \alpha_1(x^2+x) + \alpha_2(2x+1) + \alpha_3(x-1)$$

Coef. x^2 : $4 = \alpha_1$
Coef. x : $-1 = \alpha_1 + 2\alpha_2 + \alpha_3$
Const : $2 = \alpha_2 - \alpha_3$.

$(*)$ → consistent (one sol. or ∞) (Yes l.c.) or → inconsistent (No)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 2 & 1 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & -5 \end{array}\right) \longrightarrow \boxed{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & -9 \end{array}\right)}$$

since $(*)$ is consistent ...

$\Rightarrow 4x^2 - x + 2$ is a l.c. of v_1, v_2, v_3.

* If $v_1, v_2, \dots, v_n \in V$, the set of all linear combination of $v_1, v_2, \dots, v_n$ is called $\text{span}(v_1, v_2, \dots, v_n)$.

$$\text{Span}(v_1, \dots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_i \text{ scalars}\}.$$

<u>Th.</u> $\text{Span}(v_1, \dots, v_n)$ is a subspace of V.

<u>Proof:</u> $S = \text{Span}(v_1, \dots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_i \text{ scalar}\}$

- $S \neq \phi$, $0 = 0v_1 + 0v_2 + \dots + 0v_n \in \text{Span}(v_1, \dots, v_n)$
- Let $u, w \in S$.

$$\Rightarrow u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
and
$$w = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n.$$
α_i, β_i scalars.

Now, $S \overset{?}{\ni} u + w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$
$$= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \dots + (\alpha_n + \beta_n) v_n.$$

so $u + w \in \text{Span}(v_1, \dots, v_n)$.

- let $u \in S$, α scalar.

$$\Rightarrow u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n.$$
$$\alpha u = \alpha(\alpha_1 v_1 + \dots + \alpha_n v_n) = (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + \dots + (\alpha\alpha_n) v_n$$

so $\alpha u \in S$

So $S = \text{span}(v_1, \dots, v_n)$ is a subspace of V.

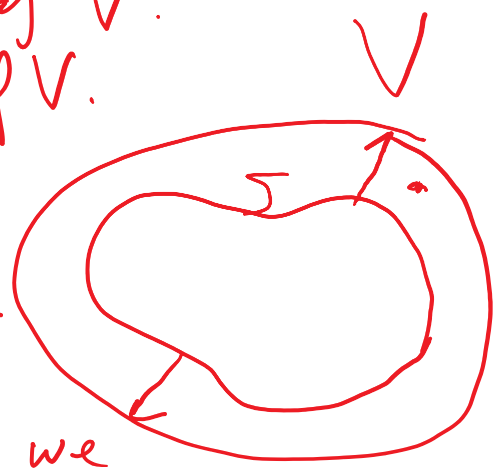
* Let V be a vector space.

1) $\{0\}$ is a subspace of V.

2) V is a subspace of V.

3) $v_1, v_2, \dots, v_n \in V$

$S = \text{span}(v_1, \dots, v_n)$ subspace of V.



If $\boxed{\text{Span}(v_1, \dots, v_n) = V}$, we say $\{v_1, \dots, v_n\}$ is a spanning set for V.

$\Longleftrightarrow$ Any vector $v \in V$ can be written as a linear combination of $v_1, \dots, v_n$.

Ex: Is $\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ a spanning set for $\mathbb{R}^3$

let $\boxed{v \in \mathbb{R}^3}$ $\quad v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

Try to solve $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3}$ for $\alpha_1, \alpha_2, \alpha_3$ — $(*)$

If $(*)$ is consistent for all $a, b, c \Longrightarrow$ $\{v_1, v_2, v_3\}$ is a sp. set for V.

If $(*)$ is consistent for some a, b, c; $(*)$ is inconsistent for some a, b, c $\Longrightarrow$ $\{v_1, v_2, v_3\}$ is not a sp. set for V.

$(*)$ $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(*) $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 1 & b \\ -1 & 1 & 0 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-2a \\ 0 & 2 & 1 & c+a \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-2a \\ 0 & 0 & 0 & c-a+b \end{array}\right)$$

Consistent $\Longleftrightarrow$ $c-a+b=0$.

Inconsistent $\Longleftrightarrow$ $c-a+b \neq 0$.

So $\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is not a sp. set.

* Ex: Is $\{ p_1(x) = x^2+x+1,\ p_2(x) = x,\ p_3(x) = 1 \}$ sp. set for P_3.

Let $p(x) = ax^2+bx+c \in P_3$.

solve: $ax^2+bx+c = \alpha_1(x^2+x+1) + \alpha_2(x) + \alpha_3(1)$

$$\left.\begin{aligned} x^2&: & a &= \alpha_1 \\ x&: & b &= \alpha_1 + \alpha_2 \\ \text{Const}&: & c &= \alpha_1 + \alpha_3 \end{aligned}\right\} \text{— } (*)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & c-a \end{array}\right)$$

(*) is consistent for all a, b, c.

So $\{ p_1(x) = x^2+x+1,\ p_2(x) = x,\ p_3(x) = 1 \}$ is a sp. set for P_3.

Ex: Is $\left\{ E = \begin{bmatrix} 1 & 0 \\ \ldots \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ \ldots \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ \ldots \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Ex: Is $\left\{ E_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}, E_2 = \begin{bmatrix}0&1\\0&1\end{bmatrix}, E_3 = \begin{bmatrix}0&0\\1&1\end{bmatrix}, E_4 = \begin{bmatrix}1&0\\0&1\end{bmatrix} \right\}$

sp. set for $\boxed{\mathbb{R}^{2\times 2}}$

Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix} \in \mathbb{R}^{2\times 2}$.

solve $\boxed{\begin{bmatrix}a&b\\c&d\end{bmatrix} = \alpha_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + \alpha_2\begin{bmatrix}0&1\\0&1\end{bmatrix} + \alpha_3\begin{bmatrix}0&0\\1&1\end{bmatrix} + \alpha_4\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_4 & \alpha_2\\ \alpha_3 & \alpha_2+\alpha_3+\alpha_4\end{bmatrix}$$

$$\boxed{\begin{aligned} a &= \alpha_1+\alpha_4\\ b &= \alpha_2\\ c &= \alpha_3\\ d &= \alpha_2+\alpha_3+\alpha_4\end{aligned}}
\qquad
\left(\begin{array}{cccc|c}1&0&0&1&a\\0&1&0&0&b\\0&0&1&0&c\\0&1&1&1&d\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c}1&0&0&1&a\\0&1&0&0&b\\0&1&1&1&d\\0&0&1&0&c\end{array}\right) \longrightarrow \left(\begin{array}{cccc|c}1&0&0&1&a\\0&1&0&0&b\\0&0&1&1&d-b\\0&0&1&0&c\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c}1&0&0&1&a\\0&1&0&0&b\\0&0&1&1&d-b\\0&0&0&-1&c-d+b\end{array}\right)$$

system is consistent for all a, b, c, d,

so $\{E_1, E_2, E_3, E_4\}$ is a sp set for $\mathbb{R}^{2\times 2}$.

⊛ m×n - system

$\circledast$ $m \times n$ - system

Given $\underset{m\times n}{A}x = b \longleftrightarrow \underline{Ax=0}$

1) * If $\boxed{z}$, w are solution to $\boxed{Ax=0}$

$A(z \pm w) = Az \pm Aw = 0 \pm 0 = 0$

* $z \pm w$ is a sol. to $Ax=0$.

* $\underline{\alpha z}$ is solution to $Ax=0$.

$(A(\alpha z) = \alpha(Az) = \alpha(0) = 0)$

$(\alpha z + \beta w)$ is a sol. to $Ax=0$.

2) If $\boxed{u, v \text{ are solutions to } Ax = b.}$

Is $u+v$ a sol. to $Ax=b$?

* $A(u+v) = Au + Av = b + b = \boxed{2b} \neq b$

$\therefore$ $u+v$ is not a solution to $Ax=b$.

* $A(u-v) = Au - Av = b - b = 0$

so $u-v$ is a sol. to $Ax=0$.

* Is $\alpha u + \beta v$ sol. to $Ax=b$

$$A(\alpha u + \beta v) = A(\alpha v) + A(\beta u) = \alpha(Av) + \beta(Au)$$
$$= \alpha b + \beta b$$
$$= (\alpha + \beta) b$$

$\boxed{\alpha u + \beta v \text{ is a sol. to } Ax=b \iff \alpha + \beta = 1.}$

$\frac{1}{4}u + \frac{3}{4}v$ is a sol. to $Ax=b$.

Th. If the system, $Ax=b$, is consistent, and

Th. If the system $Ax=b$ is consistent, and x_1 is a particular solution to $Ax=b$. Then any solution y of $Ax=b$ can be written as $y = x_1 + z$, where z is a solution to $Ax=0$. (wher $z \in N(A)$)

* x_1 sol. to $\boxed{Ax=b}$
  z sol. to $\boxed{Ax=0}$

$$A(x_1+z) = Ax_1 + Az = b + 0 = b$$

$\therefore$ x_1+z is a sol to $\boxed{Ax=b.}$

Ex: 15) 3.2 $\quad A_{4\times 3}$, $\boxed{c = 2a_1 + a_2 + a_3}$

a) If $N(A) = \{0\} \longrightarrow Ax=c$?

b) If $N(A) \neq \{0\}$, how many sol. $Ax=c$.

* $Ax=c$ is consistent. (has at least one sol.)

$x_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a sol to $Ax=c$.

all solution to $Ax=c$ have the form

$\boxed{x = x_1 + z}$, $z \in N(A)$.

(a) If $N(A) = \{0\} \longrightarrow x = x_1 + 0 = x_1$ is the only sol. to $\boxed{Ax=c}$

$\therefore$ $Ax=c$ has only one solution.

(b) If $\boxed{N(A) \neq \{0\}}$

$\therefore Ax = C$ has only one solution.

(b) If $\boxed{N(A) \neq \{0\}}$

$\Rightarrow x = x_1 + z$, $z \in N(A)$

So $Ax = C$ has infinite # of solutions.