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$\boxed{3.3}$

Let $v_1, v_2, \ldots, v_n \in V$.

consider $\boxed{c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0}$.

1) we say $v_1, v_2, \ldots, v_n$ are called <u>linearly independent</u> if the system $\boxed{c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0}$ has only the zero solution.

2) If $\boxed{c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0}$ has nonzero solution, we say $v_1, v_2, \ldots, v_n$ are <u>linearly dependent</u>

* $v_1, v_2, \ldots, v_n$ are L.D. $\iff$ there exists scalars $\boxed{c_1, c_2, \ldots, c_n}$ not all zero such that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$.

<u>Ex:</u> $v_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$. <u>L.D or L.I.</u>

solve $\boxed{c_1\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$: (*)

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -1 & 3 & 3 & 0 \\ 2 & 1 & 8 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \\ 2 & 1 & 8 \end{pmatrix}$

$\det(A) = 0$

A <u>singular</u>

$\therefore$ (*) has nonzero solutions.

... nonzero solutions.

so (*) has nonzero solutions.

$\Longrightarrow$ v_1, v_2, v_3 are L.D.

* Ex: $p_1(x) = x^2+x+1$, $p_2(x) = x-1$, $p_3(x) = x^2+1$. L.D. or L.I.

solve $\boxed{c_1(x^2+x+1)+c_2(x-1)+c_3(x^2+1)=0.}$

coef. x^2 : $c_1 + c_3 = 0$

x : $c_1 + c_2 = 0$

cons. : $c_1 - c_2 + c_3 = 0$

(*) $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1 + (-2) = -1 \neq 0$$

$\Longrightarrow$ A is nonsingular. $\Longrightarrow$ (*) has only the zero solution $c_1 = c_2 = c_3 = 0$.

$\Longrightarrow$ $\boxed{p_1(x), p_2(x), p_3(x) \text{ are L.I.}}$

Remarks: If $v_1, v_2, \dots, v_n \in \mathbb{R}^n$, then

$v_1, v_2, \dots, v_n$ are L.I. $\Longleftrightarrow$ the matrix $A = (v_1\ v_2\ \cdots\ v_n)$ is nonsingular.

* The system $\boxed{c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0}$ $\Longleftrightarrow$ $\boxed{Ac = 0}$: $A = (v_1\ v_2\ \cdots\ v_n)$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

so $\boxed{v_1, v_2, \ldots, v_n}$ are L.I $\Longleftrightarrow$ $\boxed{Ax=0}$ has only the zero solution.

columns of A.

$\Longleftrightarrow$ $\det(A) \neq 0$

$\Longleftrightarrow$ A is nonsingular

* $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ are L.D $\Longleftrightarrow$ the matrix $A = (v_1\ v_2 \ldots v_n)$ is singular.

$\Longleftrightarrow$ $\det(A) = 0$.

* The matrix $A_{n\times n}$ is nonsingular $\Longleftrightarrow$ columns of A are L.I.

Ex: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

are L.I or L.D.

Method I: solve $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

4 vectors in $\mathbb{R}^4$

Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$= 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ A is nonsingular

$\Rightarrow$ A is non-singular.

$\Rightarrow$ v_1, v_2, v_3, v_4 are L.I.

* Let $v_1, v_2, \dots, v_n \in V$ are L.D $\Leftrightarrow$ $\exists$ $\boxed{c_1, c_2, \dots, c_n \text{ not all zero}}$ such that $\boxed{c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0}$.

$c_1, c_2, \dots, c_n$ are not all zero $\Rightarrow$ at least one of $c_1, \dots, c_n$ is not zero

assume $\boxed{c_1 \neq 0}$ $\left(\boxed{\frac{1}{c_1} \in \mathbb{R}}\right)$.

$\Rightarrow$ $c_1 v_1 = -c_2 v_2 - c_3 v_3 - \dots - c_n v_n$

$\Rightarrow$ $\boxed{v_1 = -\frac{c_2}{c_1} v_2 - \frac{c_3}{c_1} v_3 - \dots - \frac{c_n}{c_1} v_n}$

$\Rightarrow$ $\boxed{v_1}$ is a linear combination of $v_2, \dots, v_n$.

* Th. Let $v_1, \dots, v_n \in V$. Then

(1) $v_1, v_2, \dots, v_n$ are L.D. $\Leftrightarrow$ one of them can be written as a linear combination of the other vectors.

(2) $v_1, v_2, \dots, v_n$ are L.I. $\Leftrightarrow$ none of them can be written as a l.c. of the other $(n-1)$ vectors.

the other $(n-1)$ vectors.

$v_1, \dots, v_n$ are L.I.

$\boxed{c_1} v_1 + c_2 v_2 + \dots + c_n v_n = 0$ has only the zero sol.

Ex: $P_1(x) = x^2 + x$, $P_2(x) = x - 1$, $P_3(x) = x^2 - 1$, $P_4(x) = 2x + 3$.

in P_3: L.I or L.D.

solve $c_1(x^2+x) + c_2(x-1) + c_3(x^2-1) + c_4(2x+3) = 0$.

Coef:

$$\begin{cases} x^2: & c_1 + c_3 = 0 \\ x: & c_1 + c_2 + 2c_4 = 0 \\ \text{cons.}: & -c_2 - c_3 + 3c_4 = 0 \end{cases} \quad (*)$$

3×4 system.

underdetermined homog. system.

$\Longrightarrow$ it has inf # of sol. (it has nonzero solutions) $\Longrightarrow$ $\boxed{P_1(x), \dots, P_4(x) \text{ are L.D.}}$

$\Longrightarrow$ one of them can be written as a l.c. of the others. (which one?)

solve $(*)$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -2 & 5 & 0 \end{array}\right)$$

$c_4 = \alpha$ free

$$\boxed{c_3 = \frac{5}{2}\alpha}$$

$$c_2 = c_3 - 2c_4 = \frac{5}{2}\alpha - 2\alpha = \frac{\alpha}{2}$$

$$c_1 = -c_3 = -\frac{5}{2}\alpha.$$

solution $= \begin{pmatrix} -\frac{5}{2}\alpha \\ \alpha/2 \\ 5/2\,\alpha \\ \alpha \end{pmatrix}$, a sol $(\alpha = 2)$ $c = \begin{pmatrix} -5 \\ 1 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -\frac{5}{2}\alpha \\ 0 \\ 5/2\,\alpha \\ \alpha \end{pmatrix}$$

If $f_1(x), f_2(x), \dots, f_n(x)$ are functions that have derivatives up to $(n-1)$-th der. we define Wronskian of $f_1(x), \dots, f_n(x)$ as

$$W[f_1(x), \dots, f_n(x)] = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \end{vmatrix}$$

$$\left.\cdots\right] = \begin{vmatrix} f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}_{n \times n}$$

Ex: $W[\sin x, \cos x]$, $\quad f_1(x) = \sin x$

$f_2(x) = \cos x.$

$$W[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x.$$

$$= -1$$