3.3

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$$W[f_1, f_2, \dots, f_n](x) = \begin{vmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & & \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

$x \in \mathrm{Dom}(f_1, \dots, f_n)$

Ex: $f_1(x) = x^2 + x$, $f_2(x) = x^3 - 1$, $\boxed{x \in [0,1]}$

$$W[f_1(x), f_2(x)](x) = \begin{vmatrix} x^2+x & x^3-1 \\ 2x+1 & 3x^2 \end{vmatrix} = \underbrace{3x^2(x^2+x) - (2x+1)(x^3-1)}_{x \in [0,1]}$$

$$W[f_1(x), f_2(x)](1) = 6 - 0 = 6.$$

(Test) Th. Let $f_1(x), f_2(x), \dots, f_n(x) \in \boxed{C^{(n-1)}[a,b]}$.
If there exists $x_0 \in [a,b]$ such that $W[f_1(x), \dots, f_n(x)](x_0) \neq 0$, then $f_1(x), \dots, f_n(x)$ are linearly independent.

* If $W[f_1(x), \dots, f_n(x)](x) = 0$, $\forall x \in [a,b]$.
Test fails ($f_1, \dots, f_n$ may be L.I or L.D)

Ex: $f_1(x) = e^x$, $f_2(x) = e^{-x} + x$. $x \in (-\infty, \infty)$.
L.I or L.D.

$$W[f_1, f_2] = \begin{vmatrix} e^x & e^{-x}+x \\ \end{vmatrix} \quad e^x(-e^{-x}+1) - e^x(e^{-x}+x)$$

$$W[f_1,f_2](x)=\begin{vmatrix} e^x & e^{-x}+x \\ e^x & -e^{-x}+1 \end{vmatrix} = e^x(-e^{-x}+1)-e^x(e^{-x}+x)$$
$$= -1+e^x-1-xe^x$$
$$= e^x - xe^x - 2.$$

$$W[f_1,f_2](0) = 1-2 = -1 \neq 0.$$
$\in(-\infty,\infty)$.

$(x_0 = 0 \in (-\infty,\infty))$

so $f_1(x)$, $f_2(x)$ are L.I.

Ex: $f_1(x) = x|x|$, $f_2(x) = x^2$, $x\in[-1,1]$.

$$W[f_1,f_2](x) = \begin{vmatrix} x|x| & x^2 \\ 2|x| & 2x \end{vmatrix}$$
$$= 2x^2|x| - 2x^2|x| = 0, \ \forall x\in[-1,1].$$

$$f_1(x)=\begin{cases} -x^2 & -1\le x\le 0 \\ x^2 & 0<x\le 1 \end{cases}$$
Cont.

$$f_1'(x)=\begin{cases} -2x & -1\le x\le 0 \\ 2x & 0<x\le 1 \end{cases}$$

$f_1'(0)^+ = 0$, $f_1'(0)^- = 0$

$\boxed{f_1'(x) = 2|x|}$.

No info (Test fails)

* $f_1(x) = x|x|$, $f_2(x) = x^2$, $x\in[-1,1]$.

solve: $\boxed{c_1 f_1(x) + c_2 f_2(x) = 0}$, (*) $\forall$ $x\in[-1,1]$.

for $\boxed{c_1, c_2}$

$c_1 x|x| + c_2 x^2 = \boxed{0}$, $\forall$ $x\in[-1,1]$.

* $x=-1$: $-c_1 + c_2 = 0$
* $x=1$: $c_1 + c_2 = 0$.

$\to$ $c_1 = 0$, $c_2 = 0$.

$>$ have only the zero sol.

- $x = 1$: $c_1 + c_2 = 0$. have only the zero sol.
- $x = \ldots$: $\ldots = 0$, $\vdots$, $\ldots = 0$

so the system has only the zero solution $c_1 = c_2 = 0$.

(*) has only the zero sol.

$\Longrightarrow$ $f_1(x) = x|x|$, $f_2(x) = x^2$ are L.I.

Ex: $f_1(x) = x|x|$, $f_2(x) = x^2$, $x \in [0,1]$,

L.I or L.D ?

$W[f_1, f_2](x) = \begin{vmatrix} & \\ & \end{vmatrix} = 0$, $\forall x \in [0,1]$.

Test fails

* Solve $c_1 f_1(x) + c_2 f_2(x) = 0$, $\forall x \in [0,1]$

$\boxed{c_1 x|x| + c_2 x^2 = 0, \quad \forall x \in [0,1].}$

- $x = \frac{1}{2}$: $\frac{c_1}{4} + \frac{c_2}{4} = 0$.
- $x = 1$: $c_1 + c_2 = 0$

have nonzero sol.

- $x = \frac{1}{4}$ $c_1 + c_2 = 0$.

$\vdots$

$\boxed{c_1 + c_2 = 0}$

$c_1 = 1$, $c_2 = -1$.

(*) has nonzero solutions. $\Longrightarrow$ $f_1(x) = x|x|$, $f_2(x) = x^2$ are L.D.

Ex: $f_1(x) = x|x|$, $f_2(x) = x^2$, $x \in [0,1]$; L.I or L.D.

$\boxed{f_1(x) = x^2}$, $\boxed{f_2(x) = x^2}$

are $\boxed{\text{L.D.}}$

$f_2(x) = 1 \cdot f_1(x)$.

$\boxed{-f_1(x) + f_2(x) = 0}$, $c_1 = -1$, $c_2 = 1$.

* $v_1, v_2, \ldots, v_n \in V$ are L.D. $\iff$

- The system $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has nonzero solution.

$\iff$ There exist $\boxed{c_1, c_2, \ldots, c_n \text{ not all zero}}$, such that $c_1 v_1 + \cdots + c_n v_n = 0$.

$\iff$ One of $\boxed{v_1, \ldots, v_n}$ can be written as a l.c. of the other $(n-1)$-vectors.

Remark! If $v_1, v_2 \in V$

v_1, v_2 are L.D. $\iff$ one of them $\boxed{\text{is a scalar}}$ multiple of the other.

Ex: $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \in \mathbb{R}^2$.

$v_2 = 2 v_1$ $\Rightarrow$ v_1, v_2 are L.D.

Ex: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ are L.I.

Ex:

Ex: $\underbrace{P_1(x) = x^2+x+1}$, $P_2(x) = x-1$.

are L.I in P_3

Ex: $\underbrace{P_1(x) = x, \quad P_2(x) = x^2}$. L.I.

$\boxed{\text{L.D}: X}$ $\quad P_2(x) = \cancel{x} \cdot P_1(x)$.

$\boxed{\text{scalar.}}$

$v_1 = \begin{pmatrix} \\ \end{pmatrix}, \ldots, v_n = \begin{pmatrix} \\ \end{pmatrix}$, $\mathbb{R}^{\boxed{n}}$

$A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$

nonsingular $\Longleftrightarrow$ columns of A are L.I.

$\boxed{A_{n\times n} \text{ nonsingular}}$

1) $Ax=0$ has only the zero solution.

2) $A \cong I$

3) $\det(A) \neq 0$.

4) $Ax=b$, has a unique solution for all $b \in \mathbb{R}^n$

5) columns of A are L.I.

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 solution for all b∈ℝ

If $v_1, v_2, \dots, v_n \in V$ - sp. set for V
- L.I or L.D.

Th. If $\{v_1, v_2, \dots, v_n\}$ is a sp. set for V and $v_1, \dots, v_n$ are L.D,

$\Rightarrow$ one of them can be written as a l.c. of the other vectors.

so the other (n-1)-vectors is a spanning set for V. $\{v_1, \dots, v_n\}$ sp. set for V. v_1 can be written as a l.c. of the other vectors $(v_2, \dots, v_n)$.

$\Rightarrow \{v_2, v_3, \dots, v_n\}$ is a sp. set for V.