× If $\{v_1, v_2, \dots, v_n\}$ is a sp. set for V, and one of them can be written as a l.c. of the other n-1 vectors, then these (n-1)-vectors is a sp. set for V.

★ one of $v_1, \dots, v_n$ can be written as a l.c. of the other (n-1)-vectors $\Longleftrightarrow$ $v_1, \dots, v_n$ are linearly dependent $\Longleftrightarrow$ there exist scalars $c_1, c_2, \dots, c_n$ not all zero s.t.

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0.$$

Def If $\{v_1, \dots, v_n\}$ is a sp. set for V and $\{v_1, \dots, v_n\}$ linearly independent, we say $\{v_1, \dots, v_n\}$ is a basis for V.

Ex: $\mathbb{R}^2$, $\left\{e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

1) L.I: $X = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $|X| = 1 \neq 0$

$\Longrightarrow \{e_1, e_2\}$ are L.I.

2) sp set for $\mathbb{R}^2$

let $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$, solve $\begin{pmatrix}a\\b\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\end{pmatrix}$

is consistent $\alpha_1 = a$, $\alpha_2 = b$.

$\therefore \{e_1, e_2\}$ is a basis for $\mathbb{R}^2$

Ex② $\left\{e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \dots, e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^n$

1) $X = \begin{pmatrix}1 & 0 & \cdots & 0\\0 & 1 & & 0\\\vdots & 0 & \ddots & \vdots\\0 & \vdots & & 0\\ & 0 & & 1\end{pmatrix} = I$ nonsingular.

$\Longrightarrow \{e_1, \dots, e_n\}$ are L.I.

2) sp set for $\boxed{\mathbb{R}^n}$

let $\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} \in \mathbb{R}^n$, solve $\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + \dots + \alpha_n \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

is consistent for all $x_1, \dots, x_n$.

Ex: Is $\left\{v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ basis for $\boxed{\mathbb{R}^3}$.

1) L.I (✓) check.

2) sp set for $\mathbb{R}^3$ (✓) check.

1) L.I (✓) check.
2) sp. set for $\mathbb{R}^3$ (✓) check.

$\boxed{\{e_1, e_2, e_3\}}$, $\boxed{\{v_1, v_2, v_3\}}$ basis for $\boxed{\mathbb{R}^3}$.

Ex: $\boxed{\mathbb{R}^{2\times 2}}$ $\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is a basis for $\mathbb{R}^{2\times 2}$.

① L.I and ② sp. set for $\mathbb{R}^{2\times 2}$ (check)

Ex: Is $\{p_1(x) = x^2, p_2(x) = x, p_3(x) = 1\}$ a basis for P_3.

1) L.I: solve $c_1 x^2 + c_2 x + c_3 = 0$

coef: x^2: $c_1 = 0$

" x : $c_2 = 0$

cond : $c_3 = 0$

so $\{x^2, x, 1\}$ are L.I.

2) sp. set for P_3, Let $p(x) = ax^2 + bx + c$.

solve $ax^2 + bx + c = \alpha_1 x^2 + \alpha_2 x + \alpha_3 (1)$.

consistent, $\boxed{\alpha_1 = a, \ \alpha_2 = b, \ \alpha_3 = c.}$

So $\{x^2, x, 1\}$ is a basis for P_3.

Th. Let $\{v_1, \dots, v_n\}$ is a spanning set for V and $\boxed{w_1, w_2, \dots, w_m} \in V$. If $\boxed{m > n}$, then $\boxed{\{w_1, \dots, w_m\}}$ are Linearly dependent.

* Ex: $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a sp. set for $\mathbb{R}^3$,

$\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \right\}$ in $\boxed{\mathbb{R}^3}$.

are L.D. (4 vectors)

* Any set of $\boxed{\text{more than } 3}$ vectors in $\mathbb{R}^3$ is L.D.

* $\mathbb{R}^n$. Any set of $\boxed{\text{more than } n}$ vectors is L.D.

$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ is a sp. set for $\mathbb{R}^n$.

Th. If $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_s\}$ are both is a sp. set for $\mathbb{R}^{\cdot\cdot}$

Th. If $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_s\}$ are both bases for V, then $n = s$.

Proof: Assume $\{v_1, v_2, \dots, v_n\}$, $\{u_1, u_2, \dots, u_s\}$ are bases for V.

* since $\{v_1, v_2, \dots, v_n\}$ is a sp. set for V and $\{u_1, u_2, \dots, u_s\}$ are L.I $\Rightarrow \boxed{s \leq n.}$ — (1)

* since $\{u_1, u_2, \dots, u_s\}$ is a sp. set for V and $\{v_1, \dots, v_n\}$ are L.I $\Rightarrow \boxed{n \leq s}$ — (2)

(1) & (2) $\Rightarrow \boxed{n = s}$

Def: Let V be a vector space, If V has a basis $\{v_1, v_2, \dots, v_n\}$, we say dimension of V = $\boxed{\dim(V) = n}$ $\{$ # of elements in any basis $\}$

* $\dim(\{0\}) := 0$, $\{$ $\{0\}$ has no basis $\}$

* If $V \neq \{0\}$ has no basis, we say V is infinite dimensional $(\dim(V) = \infty)$.

Ex: $\dim(\mathbb{R}^3) = 3$. $\{e_1, e_2, e_3\}$ basis for $\mathbb{R}^3$.

$\dim(\mathbb{R}^n) = n$ $\{e_1, e_2, \dots, e_n\}$ " " $\mathbb{R}^n$.

$\dim(\mathbb{R}^{2\times 2}) = 4$ $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ basis for $\mathbb{R}^{2\times 2}$.

$\dim(P_n) = n$

$\dim(\mathbb{R}^{m\times n}) = mn$ $\left\{ E_{ij} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & (1)_{ij} & 0 \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & \cdots & \end{pmatrix}_{m\times n} \right\}$

Ex. $C[a, b]$ has no basis

$\dim(C[a,b]) = \infty$.

Ex: Find a basis and dimension of $N(A)$,

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

$$A \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix} \xrightarrow{\frac{2}{3}R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2/3 & 3 \end{pmatrix}$$

$$A \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix} \xrightarrow{\ldots} \begin{pmatrix} \ldots \\ 0 & -3 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 3 \end{pmatrix}$$

$x_4 = \alpha$ free

$\frac{2}{3} x_3 = -3\alpha \longrightarrow x_3 = -\frac{9}{2}\alpha.$

$+3x_2 = +\left(-\frac{9}{2}\alpha\right) \longrightarrow x_2 = -\frac{3}{2}\alpha.$

$x_1 = -2\left(-\frac{3}{2}\alpha\right) - \frac{9}{2}\alpha - \alpha = 3\alpha - \frac{9}{2}\alpha - \alpha = -\frac{5}{2}\alpha$

$$\underline{N(A)} = \left\{ \begin{pmatrix} -\frac{5}{2}\alpha \\ -\frac{3}{2}\alpha \\ -\frac{9}{2}\alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalar} \right\}.$$

$$\left\{ \alpha \boxed{\begin{pmatrix} -5/2 \\ -3/2 \\ -9/2 \\ 1 \end{pmatrix}} : \alpha \text{ scalar} \right\}.$$

$\Rightarrow \left\{ v = \begin{pmatrix} 5/2 \\ 3/2 \\ 9/2 \\ 1 \end{pmatrix} \right\}$ is a sp. set for $N(A)$. and it is $\boxed{\text{L.I.}}$

so $\left\{ \begin{pmatrix} -5/2 \\ -3/2 \\ -9/2 \\ 1 \end{pmatrix} \right\}$ is a basis for $N(A)$.

$\therefore \dim(N(A)) = 1.$

Theorem: If V is a vector space, $\boxed{\dim(V) = n} > 0$

1) If $\boxed{\{w_1, w_2, \ldots, w_n\}}$ are $\boxed{\text{Linearly independent}}$ then $\{w_1, \ldots, w_n\}$ is a spanning set for V, and so a basis for V.

2) If $\{u_1, u_2, \ldots, u_n\}$ is a sp. set for V, then $\{u_1, \ldots, u_n\}$ are L.I. and so a basis.

Ex: Is $\boxed{\left\{ v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}}$ basis for $\mathbb{R}^3$.

& we know $\dim(\mathbb{R}^3) = 3$.

$\boxed{\text{L.I}}$ $\quad$ $\boxed{\text{sp. set.}}$

Let $X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\quad \det(X) = -1 - 1(-1) + (-1) = -1 \neq 0$

so v_1, v_2, v_3 are L.I.

so v_1, v_2, v_3 is a sp. set for $\boxed{\mathbb{R}^3}$.

so $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Th If V is a vector space, $\dim(V) = n > 0$.

so $\{v_1, v_2, v_3\}$ ...

Th. If V is a vector space, $\dim(V) = n > 0$.

1) If $v_1, v_2, \dots, v_s \in V$, $s < n$, then $\{v_1, \dots, v_s\}$ is not a sp. set for V.
(Any set of fewer than n vectors can't be a sp. set for V).

2) If $v_1, v_2, \dots, v_t \in V$, $t > n$, then $v_1, v_2, \dots, v_t$ are L.D.

3) If $v_1, v_2, \dots, v_r$ are L.I. and $\boxed{r < n}$ then $v_1, v_2, \dots, v_r$ can be extended to a basis.
$\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ basis

4) If $w_1, w_2, \dots, w_s \in V$, and $\{w_1, w_2, \dots, w_s\}$ is a sp. set for V and $s > n$, then $\{w_1, \dots, w_s\}$ can be paired down to a basis for V.

$\dim(\quad) = \#$ families

$\dim = \boxed{10}$

$\# < 10$ (not sp. set.)

$\#$ $w_1, \dots, w_s$, $\boxed{s > 10}$

L.D.

$\#$ $w_1, \dots, w_t$, L.I., ($t < 10$)

$\Longrightarrow$ $w_1, \dots, w_t$ can be extended to a basis

$\boxed{\begin{array}{l} * \text{ If } \underline{v_1, v_2, \dots, v_n \in V}, \ \underline{\dim(V) = n}. \\ \text{Is } \{v_1, \dots, v_n\} \text{ are } \underline{\text{L.I.}} \ (\text{not ness.}) \end{array}}$

* If $\boxed{v_1, v_2, v_3} \in V$, $\dim(V) = 5$.

$\Rightarrow$ (1) $\boxed{v_1, v_2, v_3 \ \text{L.I}}$? No info.

may or may not be L.I.

(2) v_1, v_2, v_3 sp. set for V?

<u>(not sp. set for V)</u> $(3 < 5)$

* If $\underbrace{v_1, v_2, v_3, v_4, v_5 \in V}$, $\underline{\dim(V) = 4}$.

. Is $v_1, v_2, \dots, v_5$ are <u>L.D</u> $(5 > 4)$

. Is $v_1, v_2, \dots, v_5$ <u>sp. set for V</u>?

No info. (may or may not be <u>sp. set for V</u>).