30-4

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Ex: - $\mathbb{R}^n$: basis $\left\{ e_1 = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}, \dots, e_n = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix} \right\}$

Standard basis.

- $\mathbb{R}^3$: $\left\{ e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ basis

Let $\begin{pmatrix}3\\4\\5\end{pmatrix} \in \mathbb{R}^3 \Longrightarrow \begin{pmatrix}3\\4\\5\end{pmatrix}$ can be written as a l.c. of e_1, e_2, e_3:

$$\begin{pmatrix}3\\4\\5\end{pmatrix} = 3e_1 + 4e_2 + 5e_3.$$

- P_n : standard basis

$\{1, x, x^2, \dots, x^{n-1}\}$.

$p(x) \in P_n$: $p(x) = a_0(1) + a_1 x + \dots + a_{n-1} x^{n-1}$

* $B = \left\{ v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}$ basis for $\mathbb{R}^3$

L.I: $X = \begin{pmatrix}1 & 1 & 1\\1 & -1 & 0\\1 & 0 & -1\end{pmatrix}$, $|X| \neq 0 \Longrightarrow v_1, v_2, v_3$ are L.I.

$\Longrightarrow$ a basis.

Let $\begin{pmatrix}3\\4\\5\end{pmatrix} \in \mathbb{R}^3$: solve $\boxed{\begin{pmatrix}3\\4\\5\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\-1\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}1\\0\\-1\end{pmatrix}}$

Consistent? — v_1, v_2, v_3 sp. set for $\mathbb{R}^3$

How many sol. — one sol.? — $\boxed{v_1, v_2, v_3 \text{ are L.I.}}$

Th. Let $v_1, v_2, \dots, v_k \in V$ and $\boxed{v \in \text{Span}(v_1, \dots, v_k)}$, then v can be written uniquely as a linear combination of $v_1, \dots, v_k$ iff $v_1, v_2, \dots, v_k$ are linearly independent.

* $v \in \text{Span}(v_1, \dots, v_k) \Longrightarrow v$ can be written as a l.c. of $v_1, \dots, v_k$. $\Longrightarrow \exists\ c_1, c_2, \dots, c_k$ scalars s.t. $\boxed{v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k}$ consistent.

assume $v_1, \dots, v_k$ are L.I. (show $v = c_1 v_1 + \dots + c_k v_k$ has unique solution) by contradiction assume there exists more than one solution. so $\exists$ $c_1, \dots, c_k$ and $d_1, \dots, d_k$ s.t. (not same solution)

$\boxed{v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k}$ $\boxed{v = d_1 v_1 + d_2 v_2 + \dots + d_k v_k}$

$0 = v - v = c_1 v_1 + \dots + c_k v_k - d_1 v_1 - d_2 v_2 - \dots - d_k v_k$

$\boxed{0} = (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \dots + (c_k - d_k) v_k.$

but $v_1, v_2, \dots, v_k$ are L. I. $\Longrightarrow$

$c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \dots, \quad c_k - d_k = 0$

$\Rightarrow c_1 = d_1, \quad c_2 = d_2, \dots, c_k = d_k$

$c_1, \dots, c_k$ and $d_1, \dots, d_k$ same solution.

so the system $v_1 = c_1 v_1 + \dots + c_k v_k$ has only one solution.

$$v = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 4 \\ 1 & 0 & -1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -1 & 1 \\ 0 & +1 & +2 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 3 & -3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} \rightarrow \alpha_1 = 3 - 0 - (-1) = 4. \\ \rightarrow \alpha_2 = -2 - 2(-1) = 0 \\ \rightarrow \alpha_3 = -1 \end{array}$$

$$\text{solution} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}.$$

so $$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 4 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1} + 0 \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{v_2} + (-1) \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_3}.$$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 3 e_1 + 4 e_2 + 5 e_3.$$

Basis $\{e_3, e_1, e_2\}$ for $\mathbb{R}^3$.

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 5 e_3 + 3 e_1 + 4 e_2$$

* ordered basis: $[v_1, v_2, \dots, v_n]$

is a basis with order.

Ex: $B = \left[v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right]$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 4 v_1 + 0 v_2 + (-1) v_3 .$$

coordinate vector of $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ with respect to the ordered basis B is $\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$.

$$\left[\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right]_B = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}.$$

Def: Let $B = [v_1, v_2, \dots, v_n]$ be an ordered basis for V, for $v \in V$, the coordinate vector of v with respect to B is

$$\boxed{[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}} \iff v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n .$$

Ex: Let $B = [x^2 - x, x, 2]$

1) show B is a basis for P_3

2) let $\boxed{p(x) = 3x^2 + 4x - 5}$, find $[p(x)]_B$.

① $B = [x^2 - x, x, 2]$ basis. for P_3.

3 vectors in P_3, $\dim(P_3) = 3$.

$\boxed{\text{Show L.I}}$:

solve $\boxed{c_1(x^2 - x) + c_2(x) + c_3(2) = 0}$

only zero solution $\Rightarrow$ L.I

$\boxed{\text{Check.}}$

$$W[x^2 - x, x, 2] = \begin{vmatrix} x^2 - x & x & 2 \\ 2x - 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x & 2 \\ 1 & 0 \end{vmatrix} = -4 \neq 0$$

$W[x^2 - x, x, 2](3) = -4 \neq 0.$

so $\{x^2 - x, x, 2\}$ are L.I.

3 L.I vectors in P_3 $(\dim(P_3) = 3)$, so

$[x^2 - x, x, 2]$ is a basis.

② $p(x) = 3x^2 + 4x - 5$, $[p(x)]_B$, $B = [x^2 - x, x, 2]$.

solve $p(x) = 3x^2 + 4x - 5 = \alpha_1(x^2 - x) + \alpha_2 x + \alpha_3 (2)$.

coef. x^2: $3 = \alpha_1 \longrightarrow \boxed{\alpha_1 = 3}$

coef. x^2 : $3 = \alpha_1 \longrightarrow \boxed{\alpha_1 = 3}$

" x : $4 = -\alpha_1 + \alpha_2 \longrightarrow \boxed{\alpha_2 = 7}$

Const. : $-5 = \alpha_3 \longrightarrow \boxed{\alpha_3 = -5}$

$$\boxed{[p(x) = 3x^2 + 4x - 5]_{B} = \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix}}$$

③ If $[q(x)]_{B} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, find $q(x)$. $B = [x^2 - x, x, 2]$

$\Longrightarrow q(x) = -1(x^2 - x) + 2(x) + 4(2)$

$= -x^2 + x + 2x + 8 = -x^2 + 3x + 8.$

Ex: Find $\left[\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}\right]_B$, where $B = [e_2, e_3, e_1]$.

$$\left[\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}\right]_B = \begin{pmatrix} -6 \\ 7 \\ 4 \end{pmatrix}.$$

$$\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} = -6\, e_2 + 7\, e_3 + 4\, e_1$$

(3.5) change of basis.

3.4) ⑦

$S = \left\{ \begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$ (check).

Find a basis and $\dim(S)$.

$\boxed{S \text{ subspace of } V, \ \dim(V) = n \Longrightarrow \dim(S) \leq n.}$

$S = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$

a sp. set for S is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

check L.I. (or not)

solve $c_1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

solve $c_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}+c_3\begin{pmatrix}0\\2\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&1&0&0\\1&-1&2&0\\0&1&0&0\\0&0&1&0\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&1&0&0\\0&\cancel{-2}&\cancel{2}&0\\0&1&0&0\\0&0&1&0\end{array}\right)$$

$$\longrightarrow\left(\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&-1&1&0\\0&0&1&0\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\end{array}\right)$$

$$\longrightarrow\left(\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)\longrightarrow\left.\begin{array}{l}c_1=0\\c_2=0\\c_3=0\end{array}\right\}$$

so $\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\\1\end{pmatrix}\right\}$ are L.I.,

so a basis for S is $\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\\1\end{pmatrix}\right\}$

$\therefore\ \dim(S)=3$.

Ex: $S=\left\{\begin{pmatrix}a+b+c\\b-c+d\\a+c-d\end{pmatrix} : a,b,c,d\in\mathbb{R}\right\}$. subspace of $\mathbb{R}^3$

Find a basis and $\dim(S)$.

$S=\left\{a\begin{pmatrix}1\\0\\1\end{pmatrix}+b\begin{pmatrix}1\\1\\0\end{pmatrix}+c\begin{pmatrix}1\\-1\\1\end{pmatrix}+d\begin{pmatrix}0\\1\\-1\end{pmatrix} : a,b,c\in\mathbb{R}\right\}$.

a sp. set for S is $\left\{v_1=\begin{pmatrix}1\\0\\1\end{pmatrix}, v_2=\begin{pmatrix}1\\1\\0\end{pmatrix}, v_3=\begin{pmatrix}1\\-1\\1\end{pmatrix}, v_4=\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$

check (L.I?) (L.D).

One of them can be w. as l.c. of the other 3 vectors

remove this one.

$\longrightarrow$ sp. set has 3 vectors

remove this

→ sp. set has 3 vectors

check L.I. or not.

solve $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$ (has nonzero sol.)

find a nonzero sol.