3.6 Row space

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3.6 Row space and Column Space of a matrix.

* Let $A_{m\times n}$

$\boxed{m \text{ rows}} \in \boxed{\mathbb{R}^{1\times n}}$

n columns in $\boxed{\mathbb{R}^m}$ vector spaces.

$\vec{a}_1 = (a_{11} \; a_{12} \; \cdots \; a_{1n})_{1\times n}$

$a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$

Def: If $A_{m\times n}$ is a matrix

1) The subspace of $\mathbb{R}^{1\times n}$ spanned by rows of A is called the row space of A. ($R(A)$)

2) The subspace of $\mathbb{R}^m$ spanned by columns of A is called the column space of A. ($C(A)$)

$\boxed{N(A)}$ null space

Remark: $\boxed{\dim(\mathbb{R}^{1\times n}) = n}$, $\dim(\mathbb{R}^m) = m$.

so $\boxed{\dim(R(A)) \leq n}$ and

$\boxed{\dim(C(A)) \leq m.}$

Row space: How to find a basis and dim of $R(A)$.

Th. If $\boxed{A}$, B are row equivalent matrices, then $R(A) = R(B)$

Method

* Given A, find U = the (reduced) row echelon form of A.

1) $R(A) = R(U)$

2) The nonzero rows of U form a basis for $R(U)$ and also for $\boxed{R(A).}$

Ex: Find a basis and dimension of $R(A)$, $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$

1) Find $U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ RREF.

So a basis for $R(A)$ is $\{(1 \; 2 \; 0 \; 3), (0 \; 0 \; 1 \; 2)\}$

$\dim(R(A)) = \boxed{2}$

So a basis for $R(A)$ is $\{ \ldots \}$

$\dim(R(A)) = \boxed{2}$

Remark: A has $\boxed{3 \text{ rows}}$, $\dim(R(A)) = 2$

$\Rightarrow$ $\boxed{3 \text{ rows of } A \text{ are L.D.}}$ $\quad \text{rank}(A) = 2$.

Def: rank of $A_{m \times n}$ is the $\dim(R(A))$.

$\text{rank}(A) := \dim(R(A))$.

$\boxed{4:20}$

Quiz.

Remark: $A_{5 \times 6}$, $\boxed{\text{rank}(A) = 3}$.

Are the rows of A L.I or L.D.

$$U = \begin{pmatrix} \text{———} \\ \text{———} \\ \text{———} \\ 0 - - 0 \\ 0 - - 0 \end{pmatrix}$$

* If $\text{rank}(A_{5 \times 6}) = 5$

$\Rightarrow$ 5 rows of A are L.I.

If $\text{rank}(A_{m \times n}) \neq m \Rightarrow$ rows of A are L.D.

If $\text{rank}(A_{m \times n}) = m \Rightarrow$ " " A are L.I.

$C(A)$: column space.

* Given $A_{m \times n}$, let U be the (Reduced) Row ech. form, then the columns of A have the same relations as columns of U.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$a_2 = 2a_1$ $\quad u_2 = 2u_1$

$u_4 = 2u_3 + 3u_1$

$a_4 = 2a_3 + 3a_1$

$a_2 = 2a_1$

$a_4 = 2a_3 + 3a_1$ $\longleftarrow$ $u_4 = 2u_3 + 3u_1$

* The columns in U that have leading one's are L.I. and so they form a basis for $C(U)$.

$\Longrightarrow$ The corresponding columns in A form a basis for $C(A)$.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$, Find a basis and dim of $C(A)$.

* Find $U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

So A basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \right\}$

$\Longrightarrow \dim(C(A)) = 2$.

Remark: $A_{m \times n} \longrightarrow U_{m \times n} =$

* $\dim(R(A)) = \#$ of nonzero rows in U.
$= \#$ of leading one's in U.
$= \dim(C(A))$.

$$\boxed{\text{So } \operatorname{rank}(A) = \dim(R(A)) = \dim(C(A)).}$$

* $\operatorname{rank}(A) = \#$ of leading variables in $Ax = 0$.

Ex: $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$

* basis for $R(A)$
$= \{ (1\ -2\ 1\ 1\ 2), (0\ 1\ 1\ 3\ 0), (0\ 0\ 0\ 0\ 1) \}$

$$\begin{matrix} 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{matrix}$$

$$\longrightarrow U = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$= \{ \ldots, (0\ 0\ 0\ 0\ 1) \}$

$\dim(R(A)) = 3.$

$\text{rank}(A) = 3.$

a basis for

$$\underline{C(A)} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 5 \end{pmatrix} \right\}$$