lec 11:-

Chapter 2:-

Determinant:-

* $A_{2\times2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ non singular $\Longleftrightarrow A \cong I$

$\rightsquigarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Case 1:-

$a_{11} \neq 0$

$$-\frac{a_{21}}{a_{11}} R_1 + R_2 \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \end{pmatrix}$$

So $A \cong I \Longleftrightarrow \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \neq 0$

and $a_{11} \neq 0$, $a_{11}a_{22} - a_{21}a_{12} \neq 0$

Case 2:-

If $a_{11} = 0$

$$\rightsquigarrow A \xrightarrow{R_1 \Longleftrightarrow R_2} \begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{pmatrix}$$

$A \cong I \Longleftrightarrow a_{21} \neq 0$ and $a_{12} \neq 0$

Same as:- $(a_{12})(a_{21}) \neq 0$

$\Longleftrightarrow -(a_{12})(a_{21}) \neq 0$

$\Longleftrightarrow 0 - a_{21}a_{12} \neq 0$

$\Longleftrightarrow a_{11}a_{22} - a_{21}a_{12} \neq 0$, $a_{11} = 0$

- So In Both Cases :-

$A_{2\times2}$ is non Singular $\iff a_{11}a_{22} - a_{12}a_{21} \neq 0$

Ex $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$2\times5 - 3\times4 = -2 \neq 0$

So A is nonsingular And A^{-1} does exist

← هذه الطريقة تجاوب فقط على $A \overset{?}{\Rightarrow}$ non singular?

Def :- $A_{2\times2}$

- We define determinant of A as

↳ $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

So A is nonsingular $\iff \det A \neq 0$

Ex: $A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ $\det A = 2\times3 - 1\times6 = 0$

So A is singular

$A_{3\times3}$ is nonsingular $\iff \det(A)$

~~$= a_{11}a_{22}a_{33} -$~~

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

- In General The determinant determines whether A is singular or not

→ let $A_{n\times n}$ we define the minor M_{ij} as the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$M_{2\times3} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

also we define cofactor A_{ij} as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$\longrightarrow A_{23} = (-1)^{2+3} M_{23} = -(a_{11}a_{32} - a_{31}a_{12})$$

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Ex: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\underline{a_{11}} A_{11} + \underline{a_{12}} A_{12} + \underline{a_{13}} A_{13}$$

This is equal to * في الصفحة السابقة

$$= a_{11} (-1)^2 \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} + a_{12} (-1)^3 \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ a_{13} (-1)^4 \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32}) + ^{-} a_{12} (a_{21} a_{33} - a_{23} a_{31})$$

$$+ a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

Same as *

$\det A_{3\times3} = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$ (1st row)

$= a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$ (2nd row)

$= a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33}$ (3rd row)

$= a_{11} A_{11} + a_{21} A_{21} + A_{31} A_{13}$ (1st Column)

$= \_\_\_\_$ (2nd Column)

$= \_\_\_\_$ (3rd Column)

• خذ أي row أو column وقم بالعملية ستعطيك نفس النتيجة في جميع الحالات

Def:- $\det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + a_{i3} A_{i3}$ (ith row)

$\qquad \text{———} \; a_{in} A_{in}$

$|A|$

(jth column)

$= a_{1j} A_{1j} + a_{2j} A_{2j} + \ldots \text{———} + a_{nj} A_{nj}$

A is nonsingular $\Leftrightarrow \det(A) \neq 0$

A is singular $\Leftrightarrow \det(A) = 0$

Ex: $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ nonsingular ?

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$= (1)\cdot(-1)^2 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 0(-1)^3 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= -1 + 0 - 0 = -1$$

$\rightsquigarrow$ A is nonsingular

$$\det(A) = 0\,A_{31} + 0\,A_{32} + 1\,A_{33}$$

$$= (0) + (0) + (-1)^6 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \quad \text{3rd row}$$

$$= -1$$

Remarks:-

1- $A_{3\times3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ There are 3 2×2 determinants

$A_{4\times4} = \begin{pmatrix} & & \\ & & \end{pmatrix}$ There are 4 (3×3) dete

$= 4 \times 3$ (2×2) determinant

$= 12$ (2×2) determinants

2- If A has a row or column of zeros then $\det(A) = 0$

3- If A has two identical rows or column then the determinant of $A = 0$

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \\ -1 & 4 & -1 & 1 \\ 1 & 5 & 1 & 3 \end{pmatrix}$$

$\det(A) = 0$

4) $\det(A^T) = \det(A)$

5) If A is triangular (upper or lower)

Then $\det(A)$ = products of elements on main diagonal

$\det(A) = a_{11} a_{22} \text{----} a_{nn}$

Ex:-

$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{pmatrix} \to (2)(-1)(2)(-2) = 8$$

$$\det A = 2 \begin{vmatrix} -1 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= 2 (-1)(-1)^2 \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= -2(-4) = 8$$

$\det(a_{11}) = a_{11}$

$|a_{11}|$

Theory :- If $A = (a_{ij})_{n\times n}$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A & i=j \\ 0 & i\neq j \end{cases}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = (1)(-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$+ (2)(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + (3)(-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= (12 - 15) + - (2)(1\times 6 - 3\times 4) + 3(1\times 5 - 2\times 4)$$

$$= -3 + 12 + 3\times -3$$

$$= 9 - 9 = 0$$

$A = (a_{ij})_{n\times n}$, find $\det(A)$

$A \xrightarrow[\text{operation}]{\text{row}} D = EA$

$\det(A) \xrightarrow{?} \det(EA)$ ؟العلاقة بينهما

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{-R_1+R_2} EA = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

$\det(EA) = \det(E)\det(A)$

A, where $\det(E) = \begin{cases} -1, & \text{if } E \text{ is Type I} \\ \alpha, & \text{if } E \text{ is Type II} \\ 1, & \text{if } E \text{ is Type III} \end{cases}$