lec 12

$$\det(EA) = \det(E)\det(A)$$

$$\det(E) = \begin{cases} -1 , & E \text{ of Type I} \\ \alpha , & E \text{ of Type II} \\ 1 , & E \text{ of Type III} \end{cases}$$

find $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -3 & -3 \end{vmatrix}$

$= \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -3 & -3 \end{vmatrix} = 0$

Proof

Note:
- If $E$ is Elementary, $E^T$ is elementary

$\det(EA) = \det(E)\det(A)$

$\det(AE) = \det((AE)^T) = \det(E^T A^T) = \det(E^T)\det(A^T)$

$= \det(E)\det(A)$

$= \det(A)\det(E)$

$\det(AB) =$

Theory :- let $A$ Be matrices Then

$$\det(AB) = \det(A)\det(B)$$

**Proof** :-

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Case (I): If $B$ is singular $\rightsquigarrow \det(B)=0$

Question 18 sec 1.5 :-

Result :- If $A$, $B$ are $n \times n$ matrices, $B$ is singular Then $AB$ is singular

• $AB$ is singular

$\hookrightarrow \det(AB) = 0 = \det(A)\, 0 = \det(A)\det(B)$

Case (II): if $B$ is nonsingular

$B$ can be written as the product of Elementary matrices

$\rightarrow$ i.e :- $B = E_k \text{———} E_2 E_1$, $E_i$ is element

$\det(AB) = \det((A E_k \text{——} E_2)E_1)$

$= \det(A E_k \text{——} E_2)\det(E_1)$

$=\det$

$= \det(A)\det(E_k) \text{———} \underbrace{\det(E_2)\det(E_1)}_{\det(E_2E_1)}$

$= \det(A)\det(E_k \text{——} E_1)$

$= \det(A)\det(B)$

$\det(AB) = \det A \det B = 0$

AB is singular By Q18

* $\det(kA) = k^n \det(A)_{n\times n}$ $\quad k\begin{vmatrix} \text{---} \\ \text{---} \\ \text{---} \end{vmatrix} = \begin{vmatrix} k\text{ ---} \\ k\text{ ---} \\ k\text{ ---} \\ \vdots \\ k\text{ ---} \end{vmatrix}$

* A is nonsingular ~~$\det(A^{-1})$~~

$\det(A^{-1})$ $\quad \det(A)$

$A^{-1}A = I$

$\det(AA^{-1}) = 1$

$\det(A)\det(A^{-1}) = 1$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}} \rightarrow \neq 0$$ Bcz A is nonsingular

$$2\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ 1 & -1 & 5 \end{vmatrix} \xrightarrow{?} \begin{vmatrix} 2 & 4 & -2 \\ 2 & 3 & 4 \\ 1 & -1 & 5 \end{vmatrix}$$

• الـ determ بضرب عدد على rows يحتوي على أحد فقط

Why? :-

$A \rightarrow EA$ Type II

$\det(E)\det(A) = \det(E)\det(A)$

$= \alpha \det(A)$

• إذا نطلع عامل مشترك ل det نطلع من واحد row

$$\begin{vmatrix} 2 & 4 & 6 \\ 3 & -1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

Ex:-

يدخل على جميع الـ rows ← $\underbrace{2}_{\text{scalar}}$ $\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{pmatrix}}_{\text{matrix}} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ -2 & 2 & 6 \end{pmatrix}$

$$2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} \neq \begin{vmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ -1 & 1 & 3 \end{vmatrix}$$

يدخل على row أو column (واحد فقط) $= \begin{vmatrix} 2 & 2 & 3 \\ 8 & 5 & 6 \\ -2 & -1 & 3 \end{vmatrix}$

- $\det(A) \neq 0 \iff$ if $A$ is nonsingular
- $\det(A) = 0 \iff$ if $A$ is singular

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- Def: let $A$ be $n \times n$-matrix
  we define adjoint of $A$ as

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T$$

$$= \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & \\ \vdots & \vdots & & \\ A_{1n} & \vdots & \cdots & A_{nn} \end{pmatrix}$$

Ex : $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

find adjoint of $A$ :- adj (A)

$A_{11} = (-1)^2 \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = +1$

$A_{12} = -1 \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = +2$

$A_{13} = +5$

$A_{21} = +1$

$A_{22} = -2$

$A_{23} = -3$

$A_{31} = +1$

$A_{32} = 2$

$A_{33} = 1$

$\text{adj}(A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 5 & -3 & 1 \end{pmatrix}$

$A \, \text{adj}(A) =$

$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 5 & -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I$

Zeros

$A \, \text{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a & \cdots & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & \cdots & \\ A_{1n} & & \cdots & A_{nn} \end{pmatrix}$

$$= \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ 0 & \cdots & \det(A) & 0 \\ & & \cdots & \det(A) \end{pmatrix}$$

Proof

$= \det(A)\, I$

- for any $A_{n \times n}$, $A \operatorname{adj}(A) = \det(A) I = \operatorname{adj}(A) A$

Case I :- If $A$ is nonsingular $\Rightarrow \det(A) \neq 0$

$$\frac{1}{\det(A)} A \operatorname{adj}(A) = I$$

$$A \frac{1}{\cancel{\operatorname{adj}(A)}\ \det(A)} \operatorname{adj}(A) = I$$

So :- $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

previous Ex:- $\det(A) = 4$ $A$ is nonsingular.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{5}{4} & \frac{-3}{4} & \frac{1}{4} \end{pmatrix}$$

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Case 2 if $A$ is singuler $\det(A) = 0$

$$A \operatorname{adj}(A) = \det(A) I$$

$$= 0_{n \times n}$$

* previous Example :-

$\det(A) = 4$

$\det(\operatorname{adj}(A)) = 4 + 8 + 4 = 16$

if $A$ is non Singuler Then $\operatorname{adj}(A)$ is nonsinguler $\det A \neq 0$

Proof

$$A \operatorname{adj}(A) = \det(A) I$$

$$\det(A \operatorname{adj}(A)) = \det(\det(A) I)$$

$$\det A \det(\operatorname{adj} A) = \det(A)^n \det(I)$$

$$\det(\operatorname{adj} A) = \det(A)^{n-1}$$

$\det(\operatorname{adj} A) \neq 0$ $\det(A) \neq 0$

$\operatorname{adj}(A)$ is nonsinguler

→ If $A$ is Singuler, Then $\operatorname{adj}(A)$ is singuler

We know $A \overset{\text{singular}}{\ } adj(A) = det(A) I = O_{n\times n}$ ( $det(A) = 0$ )

det ناخذ العرفين   $det(A)\, det(adj(A)) = det(O_{n\times n}) = 0$

$\quad 0$

Proof

$A\, adj(A) = O_{n\times n}$

If $adj(A)$ is nonsingular (assume)

$adj^{-1}(A)\, A\, adj(A) = O_{n\times n}$

$= A I = O_{n\times n}$

$\Rightarrow A = O_{n\times n}$

$\Longrightarrow adj(A) = O_{n\times n}$

$\Longrightarrow det(adj(A)) = 0 \Longrightarrow$ so $adj(A)$ singular (Contradiction)

So $adj(A)$ is singular

If $A$ is nonsingular

$adj\, A$ is nonsingular

$(adj(A))^{-1} = ?$ $\qquad A$

$adj(A^{-1})$