lec 13 :-

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

* $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$, $n \geq 1$

Important

* $A \operatorname{adj}(A) = \det(A)\, I$

* If A is nonsingular , then

① $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

② $\operatorname{adj}(A)$ is also nonsingular

Now: If A is singular :-

① $\operatorname{adj}(A)$ is singular

أحد الأسئلة ② $A \operatorname{adj}(A) = 0_{n \times n}$ → important

زي اسهل أمثلة للي تعطيه الـ zero matrix

* $A_{n \times n}$, $B_{n \times n}$, $AB = 0$

$\det(AB) = \det(0) = 0$

$\det(A) \det(B) = 0$

عدم وجود الشرط

$\det(A) = 0$ or $\det(B) = 0$

- A is singular or B is singular

بوجود هنا الشرط
تحقق النتيجة

If $AB = 0_{n\times n}$ and $A, B \neq 0_{n\times n}$.

Proof.

- Then A and B are singular

We have $AB = 0$

If B is nonsingular $\xrightarrow{B^{-1}}$ $ABB^{-1} = 0B^{-1}$

$A = 0$

But it's not zero

→ so B is singular

→ Similarly A is singular

Question 8 :- 2.3 :-

$$\det(\text{adj}(A)) = \det(A)^{n-1}$$

(10) Question 10 :-

If A is nonsingular, then adj(A) is nonsingular and $\text{adj}(A)^{-1}$

Proof

↳ $\text{adj}(A)^{-1} = \det(A^{-1})A = \text{adj}(A^{-1})$

① adj(A) is nonsingular

$A\,\text{adj}(A) = \det(A)\,I$

$A\,\text{adj}(A) = \frac{1}{\det A}\det(A)\,I$

$\left(\frac{1}{\det A}A\right)(\text{adj}(A)) = I$

$\text{adj}(A^{-1}) = \frac{1}{\det(A)}A = \det(A^{-1})A$

* عوضنا A^{-1} بدل A في

from * :- $A^{-1}\, adj(A^{-1}) = det(A^{-1})\, I$

نضرب A للطرفين

$adj(A^{-1}) = det(A^{-1})\, A$

Proof

So $adj(A)^{-1} = adj(A^{-1}) = det(A^{-1})\, A$

Exercise* If A is nonsingular, Then

$(A^{-1})^{-1} = A$

Proof

$A A^{-1} = I \rightarrow$ هذه تعني ان A^{-1} هي ال inverse لـ A , A

So $(A^{-1})^{-1} = A$

هي ال inverse لـ A^{-1}

Question 12 , 2.3

If determinant $(A) = 1$ show :-

وجود هنا الشرط بـ

$adj(adj(A)) = A$

Proof

$det(A) = 1 \longrightarrow A$ is nonsingular

عوضنا adj للطرفين من الـ *

$adj(A) \cdot adj(adj(A)) = det(adj(A))\, I$

$\Rightarrow adj(A)\, adj(adj(A)) = det(A)^{n-1} \cdot I$

$adj(A)^{-1} adj(A)\; adj(adj(A)) = I$ نضرب $adj(A)^{-1}$ ، $adj(A)^{-1}$

$adj(adj(A)) = A$ $\quad (1)(A)$

الطريقة أخرى :-

$\text{adj}(A)\,(\text{adj adj}(A)) = I$ بالتالي

اذن: $\text{adj}(A) = [\text{adj adj}(A)]^{-1}$

Question 9 :-

$A_{4\times4}$

If $\text{adj}(A) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{pmatrix}$

① $\det(\text{adj}(A))$

② $\det(A)$

③ A

$\det(\text{adj}(A)) = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{vmatrix}$

$-2R_2+R_3 =$
R_2+R_4 $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 8$

upper Triangul.

② $\det(A) \Rightarrow : \det(\text{adj}(A)) = \det(A)^3 = 8$

$= \boxed{\det(A) = 2}$

$$adj(A)^{-1} = \frac{1}{\det(A)} A$$

$$A = \det(A)\ \underline{adj(A^{-1})} = 2(\ )$$

Remember:- $(B)^{-1} = \frac{1}{\det(B)} adj(B)$

عوضنا $adj(A^{-1})$ بدل B^{-1} So $\underline{adj(A^{-1})} = \frac{1}{\det(\underline{adj(A^{-1})})} adj(\underline{adj(A^{-1})})$

Cramer's Rule:- $\quad A^{-1}b = X$ Remember

If $A_{n\times n}$ is nonsingular, Then

The unique solution of $Ax = b_{n\times 1}$

is Given By:- $X_i = \frac{\det(A_i)}{\det(A)}$ where

A_i is obtained from A By replacing

The ith Column of A By b

Ex: Solve By Cramers Rule:-

$$X_1 + X_2 + X_3 = 1$$
$$-X_1 - X_2 + 3X_3 = -1$$
$$2X_1 + X_2 - X_3 = 2$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 2 & 1 & -1 \end{pmatrix}, \quad \det(A) = -2 + 5 + 1 = 4 \neq 0$$

A is nonsingular:- $X_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix}}{4}$

$= 1$

$$X_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix}}{4} = \frac{0}{4} = 0$$

$$X_3 = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{vmatrix}}{4} = 0$$

So $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ A unique Solution

$A_{4\times3}$ matrix, $a_2 = a_3$, $b = a_1 + a_2 + a_3$

The System $Ax = b$?

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ | a sol $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$b = a_1 + 2a_2 + 0a_3$

a sol: $\hat{X} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a Solution

So it has infinite Solutions ← has infinit # of sol

$\hat{\hat{X}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

another one

Question 11 :- 1.3 , $b = a_1 + a_2 + a_3 + a_4$

$\boxed{Ax = b}$ and $A_{\boxed{3\times4}}$

↓

underdeterminant → infi. sol
or No sol

$b = a_1 + a_2 + a_3 + a_4$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ is a Solution

So $Ax = b$ is Consistent

↳ So infinit # of Solutions

Question 11:- $A_{5\times3}$

$b = a_1 + a_2 = a_2 + a_3$

$Ax = b$ → $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are Solution

- So $Ax = b$ has infinit number of Solutions

→ $b = a_1 + a_2 + \alpha(a_3 - a_1)$