lec 15:-

Chapter 3: Vector space

Def:- • A vector space is a set $V \neq \emptyset$ with two well defined operations (addition and Scalar Multiplication) : For all $V_1, V_2 \in V$ من نفس النوع الاولى (مخبه)

$V_1 + V_2 \in V$ and for all $v \in V$, α scaler (where $\alpha v \in V$)

Ex:: $\underbrace{\vec{V}_1}_{\in \mathbb{R}^2} + \underbrace{\vec{V}_2}_{\in \mathbb{R}^2} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in \mathbb{R}^2$

$\alpha \vec{V} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \in \mathbb{R}^2 \leftarrow$ 2D Plane لطيف •

• Such That :- تكملة التعريف :-

① $V_1 + V_2 = V_2 + V_1$, for all $V_1, V_2 \in V$

② $V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$, for all $V_1, V_2, V_3 \in V$

③ There exists a special element $\vec{0}$ (Zero vector) That satisfies $\vec{0} + \vec{V} = \vec{V}$ for all $v \in V$

بغض النظر عن شكل ال element

④ for every $v \in V$, There exists an element $-v \in V$ That satisfies :- $V + (-V) = \vec{0}$

• الموجود في الشرط الثالث

⑤ $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$, for all $v_1, v_2 \in V$ and α is a scalar

⑥ $(\alpha + \beta) = \alpha v + \beta v$ for all $v \in V$, α, β scalars

⑦ $(\alpha\beta)v = \alpha(\beta v)$, for all $v \in V$ α, β scalars

⑧ $1.v = v$, for all $v \in V$

• لنعرف ال space بأنها vector space يجب أن تتحقق ال 8 شروط
بالإضافة إلى الشرطين الموجودين في التعريف

Ex:- $V = \left\{ \begin{pmatrix} 1 \\ w \end{pmatrix}, w \in \mathbb{R} \right\}$

operations $\begin{pmatrix} 1 \\ w \end{pmatrix} + \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ w+x \end{pmatrix}$

$$\alpha \begin{pmatrix} 1 \\ w \end{pmatrix} = \begin{pmatrix} \alpha \\ w \end{pmatrix}$$

Is V under These operations a vector space?

Scanned by CamScanner

$$\begin{pmatrix} 2 \\ w+x \end{pmatrix} \notin V \quad \Bigg\} \quad \alpha \begin{pmatrix} 1 \\ w \end{pmatrix} = \begin{pmatrix} \alpha \\ w \end{pmatrix} \notin V$$

- It's Not a vector space

* اذا لم يتحقق شرط واحد على الأقل ، لا تكون vector space

Ex:- $V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} ; \; x, y \in \mathbb{R} \right\}$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in \mathbb{R}^2$$

$$\alpha \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \end{pmatrix} \in \mathbb{R}^2$$

① $$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

② $$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$\notin V$

③ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ ✓

$\vec{0} + \vec{V} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{V}$

④ let $\vec{V} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

we know $\vec{V} = \begin{pmatrix} -x \\ -y \end{pmatrix} \in \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- ⑤ $\alpha\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \alpha\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} \alpha(x_1 + x_2) \\ \alpha(y_1 + y_2) \end{pmatrix}$

$= \begin{pmatrix} \alpha x_1 + \alpha x_2 \\ \alpha y_1 + \alpha y_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \end{pmatrix} + \begin{pmatrix} \alpha x_2 \\ \alpha y_2 \end{pmatrix}$

$= \alpha\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \alpha\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

⑥, ⑦, ⑧ ✓

Ex:- $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$

under Operations :-

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}_{n\times 1} \in \mathbb{R}^n$$

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \in \mathbb{R}^n$$

standard operations on $\mathbb{R}^n$

Note:- $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Ex. $\mathbb{R}^{m\times n} = \{ A : A \text{ is a matrix of size } m\times n \}$

elements are real numbers

$A+B = C \in \mathbb{R}^{m\times n}$ (matrix addition)

$c_{ij} = a_{ij} + b_{ij}$

$\alpha A = (\alpha a_{ij}) \in \mathbb{R}^{m\times n}$

1) $A+B = B+A$
2) $A+B+C = A+(B+C)$
$\vdots$ $\vec{0} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n}$ $\quad \vec{0} + A = A$

$\vdots$ $\quad 1.\vec{A} = A$

8)

Positive Real numbers

Ex:- $V = \mathbb{R}^+ = \{ x \in \mathbb{R} \ , \ x > 0 \}$

$r_1 \oplus r_2 = r_1 r_2 \in \mathbb{R}^+$

$\alpha \odot r = r^\alpha$ دائماً معرفة

$\in \mathbb{R}^+$

دائماً معرفة لأن r موجبة دائماً

α سالبة أو موجبة

① :- $r_1 \oplus r_2 \overset{?}{=} r_2 + r_1$

$r_1 r_2 = r_2 r_1 = r_2 \oplus r_1$ ✓

② :- $r_1 \oplus (r_2 \oplus r_3) = r_1 \oplus (r_2 r_3)$

$= r_1 r_2 r_3$

$= (r_1 r_2) r_3 = r_1 r_2 \oplus r_3$

$= r_1 \oplus r_2 \oplus r_3$ ✓

③ :- $\vec{0}$

$\vec{0} \oplus r \overset{?}{=} r$

$0 \cdot r = 0$

y:- $\vec{0} = 1 \in \mathbb{R}^+$ ✓

• عشان تحقق ~~المحروه~~ الشروط اعتمد على الـ operation اللي لديك وليس اللي كنت تطبقها على الأعداد

$\vec{0} \oplus r = 1 \oplus r = 1 \cdot r = r$

④ let $r \in \mathbb{R}^+$

$-r = \frac{1}{r} \in \mathbb{R}^+$

$r + (-r) = r \cdot -r = r \times \frac{1}{r} = 1$ $\vec{0}$

$$\alpha \odot (\vec{r_1} \oplus \vec{r_2}) = \alpha \odot (r_1 r_2)$$
$$= (r_1 r_2)^\alpha$$
$$= r_1^\alpha r_2^\alpha$$
$$= r_1^\alpha \oplus r_2^\alpha$$
$$= \alpha \odot r_1 \oplus \alpha \odot r_2$$

هاي زائد عادية

⑥ $(\alpha + \beta) \odot r = r^{\alpha+\beta}$
$$= r^\alpha \cdot r^\beta$$
$$= r^\alpha \oplus r^\beta$$
$$= \alpha \odot r \oplus \beta \odot r$$

هاي زائد تبعة ال operation

هناك فرق بينهم

⑦ $(\alpha \beta) \odot r = r^{\alpha\beta}$
$$= (r^\alpha)^\beta$$
$$= \beta \odot r^\alpha$$
$$= \beta \odot (\alpha \odot r)$$
$$=$$

$$= \alpha \odot (\beta \odot r)$$

⑧ $1 \odot r = r^1 = r$

• Basic Examples in this Chapter:

① $\mathbb{R}^n$

② $\mathbb{R}^{m\times n}$

③ $C[a,b] = \{f(x) : f(x)$ is a Continuous function on $[a,b]\}$

under operations :-

$\sim (f+g)(x) = f(x) + g(x) \in C[a,b]$ standard operations

$\sim (\alpha f)(x) = \alpha f(x) \in C[a,b]$

1) $f+g = g+f$

2) $f+(g+h) = (f+g)+h$

3) $\vec{0}$: $0(x) = 0$, for all $x \in [a,b]$ zero function

4) $f(x) = -f(x)$

باقي الشروط تتحقق

Ex: $P_n = \{p(x) : p(x)$ is a polynomial of degree less than $n\}$

$P(x) = a_0 + a_1 x + \text{——} \; a_{n-1} X^{n-1}$

$Q(x) = b_0 + b_1 x + \text{——} \; b_{n-1} X^{n-1}$

$(P+Q)(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_{n-1}+b_{n-1}) x^{n-1} \in P_n$

$\alpha(P(x)) = \alpha a_0 + (\alpha a_1)x + \text{——} + (\alpha a_{n-1}) x^{n-1} \in P_n$