lec 17 :-

$$N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$$ subspace of $\mathbb{R}^n$

Ex :- find $N(A)$ for :-

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 3 & -2 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \rightsquigarrow$$

$x_4 = \alpha$ free variable

$x_3 = -\alpha$

$x_2 = \frac{2(-\alpha) + \alpha}{3}$

$= \frac{-\alpha}{3}$

$x_1 = \frac{-\alpha}{3} + \alpha - 2\alpha$

$= \frac{-4\alpha}{3}$

$$N(A) = \left\{ \begin{pmatrix} -\frac{4}{3}\alpha \\ -\frac{\alpha}{3} \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} -\frac{4}{3} \\ -\frac{1}{3} \\ -1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Reminder

- linear Combination of $a_1, \dots a_n$

Def:- let $V$ be a Vector space
$v_1, v_2, \dots v_n \in V$
a linear Combination of $v_1 \dots v_n$
has the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Ex:- $V = P_3$
$P_1(x) = x^2 + x$ , $P_2(x) = x - 1$
set of all linear Combinations
of $P_1(x)$, $P_2(x)$

Is $x^2 + x + 1$ a linear Comb of $P_1, P_2$?

$$x^2 + x + 1 \overset{?}{=} \alpha(x^2 + x) + \alpha_2(x - 1)$$

All linear Combinations of :-
$P_1(x)$ and $P_2(x)$ $= \{ \alpha_1 P_1(x) + \alpha_2 P_2(x) : \alpha_1, \alpha_2 \in \mathbb{R}$

$$S = \{ \alpha_1(x^2 + x) + \alpha_2(x - 1) : \alpha_1, \alpha_2 \in \mathbb{R}$$

$$= \{ \alpha_1 x^2 + (\alpha_1 + \alpha_2) x - \alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R} \}$$

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Does $x^2+x+1 \overset{?}{\in} S$?

$\alpha_1 = 1$, $\alpha_2 = -1$

$\alpha_1 + \alpha_2 \neq 1$

So $x^2+x+1 \notin S$

Def :- let $V$ be a vector space

$v_1, v_2, \ldots, v_n \in V$

The set of all linear Combinations of $v_1, \_\_ v_n$ is Called Span of $v_1, \ldots, v_n$

$$\text{Span}(v_1, \ldots, v_n) = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n : \alpha_i \overset{?}{\in} \mathbb{R} \right\}$$

Proprieties :- * Span $(v_1, \ldots, v_n)$ is a subspace of $V$

(1) $\text{Span}(v_1, \ldots, v_n) \neq \phi$

$0 \in \text{Span}(v_1, \ldots, v_n)$

$0 = 0v_1 + 0v_2 + \cdots + 0v_n$

* let $w, u \in \text{Span}(v_1, \ldots v_n)$

$\Rightarrow w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

and $u = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ $\quad \alpha_i, \beta_i \in \mathbb{R}$

Now $w+u = (\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n + \beta_n) v_n$

$= \gamma_1 v_1 + \cdots + \gamma_n v_n$

So $w+u \in \text{span}(v_1, \ldots, v_n)$

let $\omega \in \text{Span}(v_1, \dots, v_n)$, $\alpha$ scalar

$$\alpha w = \alpha(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha\alpha_1 v_1 + \dots \alpha \alpha_n v_n$$

$$\alpha w = (\alpha\alpha_1) v_1 + \dots (\alpha_n \alpha) v_n$$

So:- $\alpha w \in \text{Span}(v_1, \dots, v_n)$

Trivial subspace $\leftarrow$ $S = \{0\}$ subspace of $V$


$V$ is a subspace of $V$

if $\text{Span}(v_1, \dots, v_n) = V$ $(\text{span} \subseteq V)$
we say $\{v_1, \dots, v_n\}$ is a spanning set for $V$

Ex:- is $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

a spanning set for $\mathbb{R}^3$

Remark: $A = B \Rightarrow A \subseteq B$ and $B \subseteq A$

$\{v_1, \dots v_n\}$ is a spanning set for $V$ if and only if every element in $V$ can be written as a linear combination of $v_1, \dots v_n$

let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

Solve $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ —— $*$

sp. set iff $*$ is Consistent for all $a, b, c$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 0 & -1 & b \\ -1 & 1 & 2 & c \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 0 & -1 & b \\ -1 & 1 & 2 & c \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 3 & a+c \end{array}\right)$$

Consistent for all $a, b, C$

* $\{V_1, V_2, V_3\}$ is a spanning set for $\mathbb{R}^3$

Ex:- Is $P_1(x) = x^2+x$, $P_2(x) = x-1$ a sp. set for $P_3$

let $p(x) = ax^2+bx+C \in P_3$

↳ solve $ax^2+bx+C = \alpha_1(x^2+x) + \alpha_2(x-1)$

$x^2$: $a = \alpha_1$

$x$: $b = \alpha_1 + \alpha_2$

$1$: $C = -\alpha_2$

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 1 & 1 & b \\ 0 & -1 & C \end{array}\right) \Downarrow \left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b-a \\ 0 & -1 & C \end{array}\right) \Downarrow \left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b-a \\ 0 & 0 & b-a+C \end{array}\right)$$

Consistent iff $C+b-a=0$

$\rightsquigarrow$ ∴ $P_1(x)$, $P_2(x)$ is not a sp. set for $P_3$

$*$ Is :- $3x^2-2x+1 \in \text{Span}(P_1, P_2)$

$c+b-a = -4 \neq 0$

$\therefore\ 3x^2-2x+1 \notin \text{span}(P_1, P_2)$

Ex. Is $\left\{ V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ a sp.set for $\mathbb{R}^3$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

Solve :- $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ -1 & 0 & b \\ 1 & 0 & c \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b+a \\ 0 & -1 & -a+c \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b+a \\ 0 & 0 & b+c \end{array}\right) \Rightarrow \text{Consistent} \Leftrightarrow c+b=0$$

$V_1, V_2$ is not sp.set

So $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \text{span}(V_1, V_2) \Leftrightarrow c+b=0$

$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \in \text{Span}(V_1, V_2)$

Conclusion :- $S = \text{Span}(V_1, \dots, V_n)$ subspace of $V$ $\neq \phi$

→ if $\text{span}(V_1, \dots, V_n) = V$

$\Rightarrow$ We ~~use~~ say $V_1, \dots, V_n$ sp.set for $V$

• اذن S فيها عناصر ولكن لكي تكون sp.set يجب ان تكون جميع العناصر فيها يعني آخذ جبد ان أحل ال system يكون Consistent في جميع الحالات

Is $\left\{ E_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

a sp.set for $\mathbb{R}^{2\times 2}$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$

Solve : $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$a = \alpha_1 - \alpha_2 + \alpha_3$$
$$b = \alpha_1$$
$$c = -\alpha_1 + \alpha_2$$
$$d = -\alpha_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 0 & 0 & b \\ -1 & 1 & 0 & c \\ 0 & 0 & -1 & d \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -1 & -a+b \\ 0 & 0 & 1 & a+c \\ 0 & 0 & -1 & d \end{array}\right)$$

~~Consistent for~~

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & c+a \\ 0 & 0 & 0 & c+a+d \end{array}\right)$$

Consistent iff $d + c + a = 0$

$\therefore E_1, E_2, E_3$ is not a sp.set for $\mathbb{R}^{2\times 2}$

Ex: $V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $V_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Is This a spanning set of $\mathbb{R}^3$ ?

Solve: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$a = \alpha_1 + \alpha_2$$
$$b = -\alpha_1 + \alpha_3$$
$$c = \alpha_1 + \alpha_4$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ -1 & 0 & 1 & 0 & b \\ 1 & 0 & 0 & 1 & c \end{array}\right)$$
$$\Downarrow$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & a+b \\ 0 & -1 & 0 & 1 & c-a \end{array}\right)$$
$$\Downarrow$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & a+b \\ 0 & 0 & 1 & 1 & c+b \end{array}\right) \rightsquigarrow$$

Consistent for all $a, b, c$

$V_1, V_2, V_3, V_4$ is a sp.set for $\mathbb{R}^3$

Def:- let $V_1, V_2, \text{—} V_n \in V$

Is the system $C_1V_1 + C_2V_2 + \cdots + C_nV_n = 0$ ? -- $*$

لا يوجد علاقة بينهم

has only The zero Solution We say

$V_1, \text{—}, V_n$ are linearly independent

If $*$ has non zero solution We say:-

$V_1, V_2 \text{ -- } V_n$

are linearly dependent

يعتمدون على بعض (في علاقة بينهم)