lec18

$c_1v_1 + c_2v_2 + \text{---} + c_nv_n = 0$

- only the zero solution
  $v_1, \text{---}, v_n$
- linearly independent

nonzero solution

$v_1, \text{---}, v_n$

linearly Dependent

- $\exists\ c_1, c_2, \text{---}, c_n$ not all zero s.t. $c_1v_1 + c_2v_2 + \text{---} + c_nv_n = 0$

Ex: Are the vectors $\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}$ L.I or L.D

Solve $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right)$$

$$\hookrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\hookrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\ast$ has only the zero solution

$\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}$ is linearly Independent

Ex:- $P_1(x) = x^2+x+1$, $P_2(x) = x^2-1$, $P_3(x) = x-1$

$P_4 = 1-x^2-x$

L.I or L.D

Solve : $c_1(x^2+x+1) + c_2(x^2-1) + c_3(x-1) + c_4(1-x^2-x)$

$x^2 = c_1 + c_2 - c_4 = 0$
$x = c_1 + c_3 - c_4 = 0$
$1 : c_1 - c_2 - c_3 + c_4 = 0$

} underdeter homongu system

So it has non zero solutions

$\Rightarrow$ So P_1, P_2, P_3, P_4 are L.D

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right)$$

$C_4 = \alpha$ $\quad$ $C_1 = -\frac{2}{3}\alpha +$

$C_3 = \frac{2\alpha}{3}$

$C_2 = \frac{2}{3}\alpha$

Solution:- $\begin{pmatrix} \alpha/3 \\ \frac{2}{3}\alpha \\ \frac{2}{3}\alpha \\ \frac{3}{3}\alpha \end{pmatrix}$

$\alpha = 3$ A solution:-

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$

$1 \cdot P_1(x) + 2 P_2(x) + 2P_3(x) + 3P_4(x) = 0$

If it was zero, you can't write it using other polynomials

- If $C_i \neq 0$, then we can write V_i as a linear combination of the other $(n-1)$ vectors

$V_1, V_2, \ldots, V_n$ can we write V_1 as a linear com of $V_1, \ldots V_n$?

$V_1 = 1 \cdot V_1 + 0V_2 + \ldots + 0V_3$

Yes always.

Scanned by CamScanner

$V_1, \ldots, V_n$ can we write V_1 as a l.c of the other $(n-1)$ vectors $\{V_2, \ldots V_n\}$

→ Not always, There is Conditions

$V_1, V_2, \ldots, V_n \in V$
we can write one of them as a l.c of the other $(n-1)$ vectors. iff $V_1, \ldots V_n$ are l.d

l.d →
* iff There exists $c_1, c_2, \ldots c_n$ not all zero s.t $c_1V_1 + \ldots + c_nV_n = 0$
* iff $c_1V_1 + \ldots + c_nV_n = 0$ has a nonzero solution

- if $V_1, \ldots, V_n \in V$ and $\{V_1, V_2, \ldots V_n\}$ is a spanning set.

→ If one of them (say V_1) can be written as a l.c of the other vectors
The remaining $(n-1)$ vector (remove V_1) form a spanning set for V

Cases:-

- if $\{V_1, \ldots, V_n\}$ is a spanning set
$v \in V$

Then $\{V_1, V_2, \ldots V_n, V\}$ is a sp. set

- If I removed v_{n-1}
  - $\{v_1, \dots, v_{n-1}\}$ may or may not be a sp. set ~~linearly~~

- If $\{v_1, \dots, v_n\}$ L.I, $v \in V$
  - $\{v_1, v_2, \dots, v_n, v\}$ may ~~not~~ or may not be L.I
  - $\{v_1, \dots, v_{n-1}\}$ are L.I

- If $\{v_1, v_2, \dots, v_n\}$ are L.D, $v \in V$
  - $\{v_1, \dots, v_n, v\}$ are L.D
  - $\{v_1, \dots, v_{n-1}\}$ may or may not be L.D

functions :-

Ex: $f_1(x) = \cos x$, $f_2(x) = \sin x$

are L.I or L.D in $C[0, 2\pi]$

Solve :- $C_1 \cos x + C_2 \sin x = 0$, $\forall x \in [0, 2\pi]$

$o(x) = 0 \Rightarrow$ zero function $\forall x \in [0, 2\pi]$

Remember / $c_1(x^2+x+1) + c_2(x^2-x) = 0$; we used to solve the system

here :-

$\cos x$, $\sin x$ are L.D $\iff \exists\, c_1, c_2$ not all zero s.t $c_1 \cos x + c_2 \sin x = 0$ $\forall x \in [0, 2\pi]$

- If $f_1(x), \ldots, f_n(x) \in C^{n-1}[a,b]$

$\Downarrow$

$f_1, \ldots f_n$ have derivatives up to $n-1$ derivatives

We define Wronskian of $f_1, \ldots f_n$ as

$$W[f_1, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{n-1}(x) & & \cdots & f_n^{n-1}(x) \end{vmatrix}$$

Theory :- let $f_1(x), \ldots, f_n(x) \in C^{n-1}[a,b]$

If there exists $x \in [a,b]$ such that $W[f_1(x), \ldots f_n(x)](x_0) \neq 0$ then $f_1(x), \ldots f_n(x)$ are L.I (Test)

Ex: $f_1(x) = \sin x$
$f_2(x) = \cos x$ in $C[0, 2\pi]$

$$W[\cos x, \sin x](x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$-\sin^2 x - \cos^2 x = -1 \neq 0$$

$$W[\ \ ](\pi) = -1 \neq 0$$

$\Longrightarrow$ $\sin x, \cos x$ are L.I

Ex $f_1(x) = x^2$, $f_2(x) = x|x|$ L.I or L.D $[-1, 1]$

$\in C[-1,1]$ diff

$$W[f_1(x), f_2(x)](x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$$

$$x|x| = \begin{cases} x^2 & 0 \le x \le 1 \\ -x^2 & -1 \le x \le 0 \end{cases}$$

$$(x|x|)' = \begin{cases} 2x & 0 \le x \le 1 \\ -2x & -1 \le x \le 0 \end{cases} = 2|x|$$

$$2x^2|x| - 2x^2|x| = 0$$

Test fails