lec 20:-

$v_1, v_2, \ldots, v_n \in V$

* Spanning set for V

$\Leftrightarrow$ every $v \in V$ can be written as $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$

* linearly Independent or dependent

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

only zero sol — L.I

non zero sol ($\exists\, c_1, \ldots, c_n$ not all zero s.t $c_1 v_1 + \ldots + c_n v_n = 0$) — L.D

3.4) Basis :-

Def: let $v_1, \ldots, v_n \in V$, we say $\{v_1, \ldots, v_n\}$ is a basis for V if

1) $\{V_1, \dots, V_n\}$ is a sp set for V

2) $V_1, V_2, \dots, V_n$ are L.I

Ex:- $\left\{ V_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, V_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Is a basis for $\mathbb{R}^3$?

$\boxed{1}$ are V_1, V_2, V_3 a spanning set for V?

let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 2 & 0 & -1 & b \\ -1 & 1 & 1 & c \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 2 & -3 & b-2a \\ 0 & 0 & 2 & a+c \end{array}\right)$$

So it's Consistent for all a, b, c

$\{V_1, V_2, V_3\}$ form a sp. set for $\mathbb{R}^3$

2- Are They L.I ?

$$X = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$|X| = (1)(1) + (1) + (1)(2)$$
$$= 4 \neq 0$$ So The matrix is nonsingular and Columns of x are L.I

V_1, V_2, V_3 are linearly .I

so $\{V_1, V_2, V_3\}$ is a basis for $\mathbb{R}^3$

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Ex: is $\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right.$

a basis for $\mathbb{R}^{2\times 2}$?

i) sp-set ?

let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It's obvious That

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

So $\alpha_1 = a$ $\alpha_3 = c$

$\alpha_2 = b$ $\alpha_4 = d$

So $E_{11}, E_{12}, E_{21}, E_{22}$

form a sp set

2) L.I ?

$$c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

it's obvious That

$$\begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$c_1 = c_2 = c_3 = 0 \implies$ only zero sol

$\longrightarrow$ E_{11}, —— , E_{22} are L.I

$\rightsquigarrow$ So basis for $\mathbb{R}^{2\times 2}$

Ex:- $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ basis for $\mathbb{R}^3$

1) sp set for $\mathbb{R}^3$?

$$\begin{pmatrix}a\\b\\c\end{pmatrix}=\alpha_1\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha_2\begin{pmatrix}0\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Consistent for all a, b, c

$$= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \qquad \begin{matrix} \alpha_1 = a \\ \alpha_2 = b \\ \alpha_3 = c \end{matrix}$$

$\therefore$ e_1, e_2, e_3 form a sp.set for $\mathbb{R}^3$

L.I ?: $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $|X| = 1$

X is nonsingular

e_1, e_2, e_3 are L.I

$\Longrightarrow$ e_1, e_2, e_3 form a basis for $\mathbb{R}^3$

Ex:- IS $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, e_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$

Sp. set + L.I $\Rightarrow$ Minimal Spanning Set

1) Sp set?

Yes it is

2) L.I : It's not a square matrix so you can't use The matrix way

$$\text{Solve :- } c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}+c_4\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Underdeterminate homogeneous System $\Rightarrow$ has no zero sol

So e_1, e_2, e_3, e_4 are L.D

<u>Theory</u> :- If V has a spanning set with n vectors $\{V_1, \dots, V_n\}$ any set of more Than n vectors is L.D

If $(\omega_1, \dots, \omega_m \in V,\ m > n)$

$\Rightarrow \omega_1, \dots \omega_n$ are L.D

$c_1 w_1 + \text{———} + c_n w_n = 0$ $\left\{ \begin{array}{l} n \times m \text{ system} \\ \text{equations} \end{array} \right.$ unknown

$. m > n$
underdetermined homogeneous system

has non zero sol
$\therefore w_1, \text{—}, w_n$ are L.D

Theory : let V be a vectorspace if $\{v_1, v_2, \text{—}, v_n\}$ and $\{u_1, u_2 \text{—} u_m\}$ are basis for V, Then $\boxed{n = m}$

Proof :- $\{v_1, \text{—}, v_n\}$, $\{u_1, \text{—}, u_m\}$ are basis

Since $\{v_1, \text{—}, v_n\}$ is a sp.set for V and $\{u_1, \text{—}, u_m\}$ are L.I

Then $\boxed{m \leq n}$

Since $\{u_1, \text{—}, u_m\}$ is a sp.set for V and $\{v_1, \text{—} v_n\}$ are L.I $\boxed{n \leq m}$

So :- $\boxed{m=n}$

Def:- let V be a vector spac. If V has a basis with n vectors $\{V_1, \ldots, V_n\}$. we say dimension of V is n

- (if not, if b has no basis we say dimension of V is infinit)

$\dim(V) = n$

Note:- • $\dim\{0\} = 0$

• $\dim\{\mathbb{R}^3\} = 3$

$\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$

$\dim\{\mathbb{R}^n\} = n$

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \_\_\_, e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^n$ (standard basis)

- Does $V = \{0\}$ has a basis?

$S = \{0\}$ ← $0 = 1.0$

- S is a Spanning set for V
- $\{0\}$ L.I?

$C_1(0) = 0$ has non zero

it can be any number Solution

So $V = \{0\}$ has no basis

Ex 1) $\dim(\mathbb{R}^{2\times 2}) = 4$

2) $\dim(\mathbb{R}^{m\times n}) = mn$

basis for $\mathbb{R}^{m\times n}$:-

$$\left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & \\ \vdots & & & \\ 0 & & & \end{bmatrix}, \longrightarrow \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \; 1 \end{bmatrix} \right\}$$

$\dim(\mathbb{R}^{2\times 3}) = 6$

basis : $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Ex : $\dim(P_3)$

$B_{P_3} = \{x^2, x, 1\}$ is a basis for

1) Sp set ?

let $ax^2+bx+c = \alpha_1 X^2 + \alpha_2 X + \alpha_3(1)$

Consistent for all a, b, c

2) L.I? L.I $c_1X^2+c_2X+c_3=0$

$\left.\begin{array}{ll} c_1=0 & c_3=0 \\ c_2=0 & \end{array}\right\} \rightarrow$ L.I

So $\dim(P_3)=3$

$\longrightarrow \dim(P_n)=n$

basis : $\{X^{n-1}, X^{n-2}, \text{———}, X, 1\}$

* Ex, is $[X^2+X+1, X-1, X^2-1]$ a basis for P_3 ?

- It's a sp and L.I ~~set~~ So it is a basis

Theory : let V be a vector space, $\dim(V) = n \geq 1$
let $V_1, V_2, \ldots, V_n \in V$

1) If $V_1, \ldots, V_n$ is a spanning set for V Then $V_1, \ldots, V_n$ are L.I and so $\{V_1, \ldots, V_n\}$ is a basis

2) If $\{V_1, \ldots, V_n\}$ are L.I Then They form a spanning set for V and so form a basis

Back to *
Three vectors in P_3, $\dim(P_3) = 3$
L.I ? $C_1(X^2+X+1) + C_2(X-1) + C_3(X^2-1) = 0$

Solve :- $X^2 : C_1 + C_3 = 0$
$X : C_1 + C_2 = 0$
$1 : C_1 - C_2 - C_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

so $c_1 = c_2 = c_3 = 0$

$\Rightarrow$ L.I (3 L.I vectors in P_3)

$\Rightarrow \{ x^2 + \ldots \}$ is a basis for P_3