lec 21:-

Basis , dim(V)

If (V) has a basis $\{V_1, \ldots, V_n\}$
$\dim(V) = n$

V has no basis
$\rightarrow \dim(V) = \infty$

$V = \{0\}$, $\dim(V) = 0$
has no basis

- $\dim(V) = n$

1) if $\{V_1, \ldots, V_n\}$ L.I $\Rightarrow$ sp. set for V
(basis)

2) if $\{V_1, \ldots, V_n\}$ sp. set for $V \Rightarrow$ L.I

Ex: Is $\{P_1(x) = x^2+x+1,\ P_2(x) = x^2-1,$
$P_3(x) = x-1\}$
a basis for P_3

3 vectors in P_3 : L.I?

To know use:-

1. Wronskian or
2. Solving system :

$\hookrightarrow$ $C_1(x^2+x+1) + C_2(x^2-1) + C_3(x-1) = 0$

x^2: $C_1 + C_2 = 0$

x: $C_1 + C_3 = 0$

1: $C_1 - C_2 - C_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) \rightarrow \text{only zero sol}$$

So They are L.I

$\dim(P_3) = 3$

$\rightarrow P_1, P_2, P_3$ form a basis

Ex: Is $\left\{ V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, V_3 = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$

$\dim(\mathbb{R}^3) = 3$

نتيجة مهمة • عشان ال Dim يساوي n لذن نستطيع
نأخذ شرط واحد فقط L.I او sp.set

$$X = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & 7 \end{pmatrix} \quad L.I\,?$$

$$|X| = (-1)(5) - (1)(-5)$$
$$= 0$$

- X is singuler
- V_1, V_2, V_3 are L.D

$\Rightarrow V_1, V_2, V_3$ are not a basis

Ex: Is $\left\{ A_1 = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^{2\times 2}$

Not a basis (3 vectors in $\mathbb{R}^{2\times 2}$

$\dim(\mathbb{R}^{2\times 2}) = 4$)

Note: dim is unique

if I don't Know The dim I have to look for The two Conditions

if I Know it Then if n of elements equals dim we use L.I test

if n of elements is more or less than dim Then it's not a basis

Results :-

- $\dim(\mathbb{R}^n) = n$

$\{x^{n-1}, x^{n-2}, x, 1\}$ standard basis

- $\dim(\mathbb{R}^{m\times n}) = mn$

$$\left\{ E_{11} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \vdots & & & \vdots \\ \vdots & \cdots & & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 & \cdots \\ \cdots & & \\ \cdots & & \end{bmatrix} \cdots\cdots, E_{mn} = \begin{bmatrix} 0 & \cdots & \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right\}$$

standard basis

- $\dim(\mathbb{R}^n) = n$

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots\cdots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

standard basis

- $C[a,b]$ has no basis

$\dim(C[a,b] = \infty)$

Theory :- let V be a vector space

dim $(V) = n > 0$

1) Any set of less than n vectors can't be a sp.set

2) Any set of more than n vectors is linearly depend

3) Any linearly independent set with less than n vectors can be extended to a basis

4) Any sp.set with more than n vectors can be paired down to a basis

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Ex: — extend $\left\{ V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$ $(\in \mathbb{R}^3)$ to a basis

- are L.I

$$\begin{pmatrix} 1 & 2 & a \\ -1 & -1 & b \\ 1 & 0 & c \end{pmatrix}$$

- Choose a, b, c s.t X is nonsingular

  - You can put any value for a, b and calculate $|X|$ and choose c s. That makes $|X| \neq 0$

So,

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & c \end{pmatrix}$$

$|X| = 1 + c \Rightarrow |X| \neq 0$

$1 + c \neq 0$

$c \neq 1$

Take $c = 2$:-

$$\left\{ V_3 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$\rightarrow$ L.I , 3 vectors in $\mathbb{R}^3$

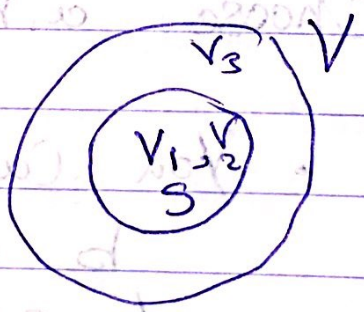
$\{V_1, V_2\}$ L.I $\quad \dim(\mathbb{R}^3) = 3$

$\text{span}(V_1, V_2) \neq \mathbb{R}^3$

not sp. set

$\rightarrow$ Choose V_3 s.t

$V_3 \notin \text{span}(V_1, V_2)$



Choose V_3 s.t

$V_3 = \alpha_1 V_1 + \alpha_2 V_2$

is inconsistant

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

is inconsistant

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ -1 & -1 & b \\ 1 & 0 & c \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & a+b \\ 0 & -2 & c-a \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & a+b \\ 0 & 0 & a+2b+c \end{array}\right)$$

Choose a, b, c s.t

$c + a + 2b \neq 0$

$c \neq -a - 2b$

Take $a = 2$ $b = 4$

$c \neq -2 - 8 = -10$

$c \neq -10$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \rightarrow c \neq 10$$

* $\{V_1, V_2, \ldots, V_s\}$ L.I

$\Rightarrow \dim(V) = n \Rightarrow s < n$

extend to basis

Choose $w_1 \notin \text{span}(V_1, \ldots, V_s)$

$\{V_1, \ldots, V_s, w_1\}$ are L.I

$\boxed{s+1}$

$\boxed{s+1 = n}$

$\{V_1, \ldots, V_s, w_1\}$ is a basis

$\boxed{s+1 < n}$

- Choose w_2 s.t

$w_2 \notin \text{span}(V_1, \ldots, V_s, w_1)$

$\{V_1, \ldots, V_s, w_1, w_2\}$ L.I

$= \boxed{s+2}$

$\boxed{s+2 = n}$

$\{V_1, \ldots, V_s, w_1, w_2\}$ is a basis

$\boxed{s+2 < n}$

Continue

form a sp. set

Ex: reduce $\left\{ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right.$

$\left. v_3 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$ to a basis $\in \mathbb{R}^2$

$\dim(\mathbb{R}^2) = 2$

- remove (1) The one That can be written as a l.c of the others

How?, Solve $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

The one That it's coefficient does not equal zero can be removed

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -2 & 0 \\ -1 & 3 & 5 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -2 & 0 \\ 0 & 5 & 9 & 1 & 0 \end{array}\right)$$

* $c_4 = \beta$, $c_3 = \alpha$ free
* $c_2 = \dfrac{-9\alpha - \beta}{5}$

$c_1 = -2\left(\dfrac{-9\alpha - \beta}{5}\right) - 4\alpha + 2\beta$

$= -\dfrac{2\alpha}{5} + \dfrac{12}{5}\beta$

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• I remove C_4 or C_3
(only one at a time)

In this Ex ; remove V_4

So $\{V_1, V_2, \_, V\}$ is a sp. set

• 3 in $\mathbb{R}^2 \Longrightarrow$ L.D

one of them can be written as a l.c of the others (remove it)

Solve $d_1 v_1 + d_2 v_2 + d_3 v_3 = 0$

$$d_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + d_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + d_3 \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 3 & 5 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 5 & 9 & 0 \end{array}\right)$$

$d_3 = \alpha$ free $\Longrightarrow$ remove V_3.

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$\Longrightarrow \left\{ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

form a sp. set for $\mathbb{R}^2$

- 2 vectors in $\mathbb{R}^2 \Rightarrow$ sp. set $\Rightarrow$ basis for $\mathbb{R}^2$

So $\left\{ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

3.4 is finished

* Back to 3.2 :-

Theory :- • let A be $m \times n$ - matrix $b \in \mathbb{R}^n$

↳ • If $Ax = b$ is Consistent and let x_0 be a particular Solution to $Ax = b$

↳ A vector y is a Solution to $Ax = b$ iff $y = x_0 + z$, with z is a Solution to $Ax = 0$

$\underline{y}$ = Sol to homoge. + Sol to nonhomog

Sol to nonhomog

$$\left\{ z \in N(A) \right\}$$