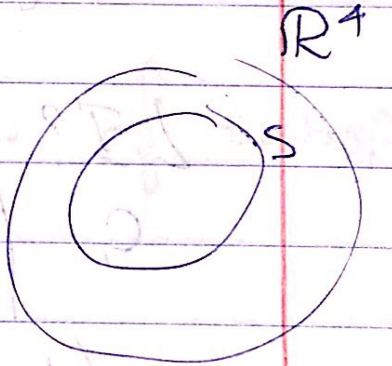
lec 22:-

$$S = \{(a+b,\ a-b+2c,\ b,\ c)^T,\ a,b,c \in \mathbb{R}$$

$$\left\{ S = \begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix},\ a,b,c \in \mathbb{R} \right\}$$

Subspace of $\mathbb{R}^4$



$\dim(S) \leq 4$

$$\begin{pmatrix} 2 \\ 4 \\ 5 \\ 7 \end{pmatrix} \notin S$$

$a - 5 + 14 = 4$

$b = 5 \quad a = -5$

$c = 7$

$a + b = 0 \neq$

$$S = \left\{ a\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\2\\0\\1\end{pmatrix},\ a,b,c \in \mathbb{R} \right\}$$

Spanning set for S is

$$\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\end{pmatrix} \right\}$$

L.I?

$$c_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & \cancel{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow$$

$c_1 = 0$
$c_2 = 0$
$c_3 = 0$

So $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\end{pmatrix}$ are L.I

So $\left\{ \sim, \sim, \sim \right.$ is a basis for S

$\dim(S) = 3$

Ex (14) (3.4)

b) find dimension of subspa spanned by

$x, \; x-1, \; x^2+1, \; x^2-1$

$$S = \mathrm{Span}\left(x, x-1, x^2+1, x^2-1 \right)$$

$$= \left\{ ax + b(x-1) + c(x^2+1) + d(x^2-1), \; a,b,c,d \in \mathbb{R} \right\}$$

Spanning set for S : $x, x-1, x^2+1, x^2-1$

S is a subspace for P_3

$\dim(S) \leq 3$

L.I ?

$c_1 x + c_2(x-1) + c_3(x^2+1) + c_4(x^2-1) = 0$

x^2 : $c_3 + c_4$

x : $c_2 + c_1$

1 : $-c_2 + c_3 - c_4$

underdetermined homog system

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

~~$x_3 + x_4 =$~~

$c_4 = \alpha$

remove c_4

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

- remove x^2-1

* $\{x, x-1, x^2+1, x^2-1\}$ form a spanning set for S

L.I?

Solve: $c_1x + c_2(x-1) + c_3(x^2+1) = 0$

$$x^2: \; c_3 \qquad \Rightarrow \begin{cases} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{cases}$$
$$x: \; c_1 + c_2$$
$$1: \; -c_2 + c_3$$

So $x, x-1, x^2+1$ are L.I

and a basis for S is

$\{x, x-1, x^2+1\}$

$\dim(S) = 3$

Note

- S subspace of P_3

$\dim(S) = 3$ $\qquad \dim(P_3) = 3$

So $S = P_3$

3.5:

$\{V_1, V_2, \dots, V_n\}$ is a basis for V

$\Longleftrightarrow$ Any vector $v \in V$ can be written Uniquely as

$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4$

- ordered Basis :- $B = [V_1, V_2, \dots, V_n]$ basis with order

Coordinats of v with respect to B as $[V]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{pmatrix}$

V_1 ⅃ Coeff of α_1 lolies.
V_2 ⅃ Coeff of α_2 ,
ξ ——— .

Ex:- if $B = \left[v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$

1) and $[V]_B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ find V

2) find coordinate vector $\left[\begin{pmatrix} 5 \\ 7 \end{pmatrix} \right]_B$

① $[V]_B = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Longrightarrow V = 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 3 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

② $\left[\begin{pmatrix} 5 \\ 7 \end{pmatrix} \right]_B = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ -\alpha_1 + 0 \end{pmatrix}$$

$$\alpha_1 = -7$$
$$\alpha_2 = -12$$

Scanned by CamScanner

$$\left[\begin{pmatrix}5\\7\end{pmatrix}\right]_B = \begin{pmatrix}-7\\-12\end{pmatrix}$$

الترتيب مهم .

$$S = \left[e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}\right]$$

find $\left[\begin{pmatrix}5\\6\end{pmatrix}\right]_S$ $\quad \begin{pmatrix}5\\6\end{pmatrix} = \alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix}$

$\alpha_1 = 5 \quad , \alpha_2 = 6$

$$\left[\begin{pmatrix}5\\6\end{pmatrix}_S\right] = \begin{pmatrix}5\\6\end{pmatrix}$$

* Change of basis

$U = [u_1, u_2, \ldots, u_n]$

$W = [w_1, w_2, w_3 \ldots, w_n]$ bases for V , $v \in V$

Given $[v]_U \longleftrightarrow$ find $[v]_W$

New basis

* Now from $[v]_{u}$ ⟶ find $[v]_{\omega}$
old basis

① Write the elements of old basis as a l.c of elements of new basis

Since $\omega_1, \dots \omega_n$ sp set for V

$$u_1 = \alpha_{11}\omega_1 + \alpha_{12}\omega_2 + \alpha_{13}\omega_3 + \dots + \alpha_{1n}\omega_n$$
$$u_2 = \alpha_{21}\omega_1 + \alpha_{22}\omega_2 + \text{———} + \alpha_{2n}\omega_n$$
$$\vdots$$
$$u_n = \alpha_{n1}\omega_1 + \alpha_{n2}\omega_2 + \text{——} \alpha_{nn}\omega_n$$

So any element in V can be written in terms of ω and $u_1 \in V$, $u_2 \dots$

Given * $[v]_u = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

$$\Rightarrow v = c_1 u_1 + c_2 u_2 + \text{— —} + c_n u_n$$

$$= c_1(\alpha_{11}\omega_1 + \alpha_{12}\omega_2 + \dots + \alpha_{1n}\omega_n)$$
$$+ \text{— —} + c_n(\alpha_{n1}\omega_1 + \text{— —} \alpha_{nn}\omega_n)$$

$$V = (c_1\alpha_{11} + c_2\alpha_{21} + \cdots + c_n\alpha_{n1})\,\omega_1$$
$$+ (c_2\alpha_{12} + c_2\alpha_{22} + \cdots + c_n\alpha_{n2})\,\omega_2$$
$$+ \cdots + (c_1\alpha_{1n} + c_2\alpha_{2n} + \cdots\ c_n\alpha_{nn})\,\omega_n$$

$$[V]_\omega = \begin{pmatrix} c_1\alpha_{11} + \cdots + c_n\alpha_{1n} \\ c_2\alpha_{12} + \cdots + c_n\alpha_{n2} \\ \vdots \\ c_1\alpha_{1n} + \cdots + c_n\alpha_{nn} \end{pmatrix}$$

$$[V]_\omega = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & & & \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{pmatrix}}_{n \times n} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{n \times 1}$$

*

$$[V]_\omega = T_{u \to w}[V]_u$$

matrix

$T_{u\to w}$: Transition Matrix from U to W

Ex: find the Transition Matrix from

$$U = \left[V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]$$

to $S = \left[e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

$$u_1 = \alpha_{11} w_1 + \alpha_{12} w_2 + \cdots + \alpha_{1n} w_n$$

$$[u_1]_w = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \end{pmatrix}$$

$$[u_2]_w = \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \end{pmatrix}$$

So $\underset{u \to w}{T} = \left([u_1]_w, [u_2]_w, \cdots, [u_n]_w \right)_{n \times n}$

$T_{u \to s}$:-

$u_1 = -1e_1 + 1e_2$
$u_2 = 2e_1 + 1e_2$

$$T_{u \to s} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} = (u_1 \; u_2)$$

Remark:- $T_{u \to s}$ $\mathbb{R}^n$ $\underline{U} = \left[u_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{pmatrix}, \ldots, u_n = \begin{pmatrix} u_{n1} \\ \vdots \\ u_{nn} \end{pmatrix} \right]$

$\underline{S} = [e_1, \ldots, e_n]$

$$T_{u \to s} = (u_1 \; u_2 \ldots u_n)_{n \times n}$$

b) find $T_{S\to u}$:-

$$e_1 = \alpha_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-\alpha_1 + 2\alpha_2 = 1$$

$$\alpha_1 + \alpha_2 = 0$$

$$-\alpha_1 + 2(-\alpha_1) = 1 \qquad \alpha_1 = -\tfrac{1}{3}$$

$$\alpha_2 = \tfrac{1}{3}$$

$$e_2 = \beta_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$-\beta_1 + 2\beta_2 = 0$$

$$\beta_1 + \beta_2 = 1$$

$$\rightarrow \beta_2 = \tfrac{1}{3} \text{ and } \beta_1 = \tfrac{2}{3}$$

$$e_1 = -\tfrac{1}{3} u_1 + \tfrac{1}{3} u_2$$

$$e_2 = \tfrac{2}{3} u_1 + \tfrac{1}{3} u_2$$

$$T_{S\to u} = \begin{pmatrix} -1/3 & 2/3 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$T_{u \to s}$:-

$$[V]_w = T_{u \to w} [V]_u$$

always non singuler (n×n)

$$(T_{u \to w})^{-1} [V]_w = [V]_u$$

So $T_{w \to u} = (T_{u \to w})^{-1}$

$$T_{s \to u} = (T_{u \to s})^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \frac{-1}{3} \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$