lec 24 :-

$A_{m\times n}$

① $R(A)$ = span(rows of A)

$\rightsquigarrow$ subspace of $\mathbb{R}^{1\times n}$

$\dim(R(A)) \leq \min\{m,n\}$

If V has a spanning set $\{v_1, \dots, v_n\}$

$\Rightarrow \dim(V) \leq n$

Two row equivalent matrices has the same row space

if $A \cong B \rightarrow R(A) = R(B)$

Given $A_{m\times n}$, let U be the R.R.E.F

The nonzero rows in U form a basis for $R(U)$ and for

$R(A)$

Ex. find a basis and dimension of $R(A)$.

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & & -1 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

basis for $R(A)$ is

$\{ (1 \ -1 \ 2 \ 1), (0 \ 1 \ -1 \ 1), (0 \ 0 \ -2 \ 1)$

$\cdot \dim R(A) = 3$

Ex: $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}$, $R(A)$

$$\rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Basis for ~~row~~ R(A) is

$$\left\{ (1 \ -1 \ 1), \ (0 \ 1 \ -1) \right\}$$

$\rightsquigarrow$ dim R(A) = 2

$\rightsquigarrow$ row of A are LD

basis for $R(A)$ is :-

$\{ (1\ -1\ 2\ 1), (1\ -1\ 0\ 2), (0\ 1\ -1\ 1) \}$

Remark:-

* If $\{v_1, \dots, v_n\}$ is a basis for V

Then $\{\alpha v_1, \alpha v_2, \dots, \alpha v_n\}$

is a basis for V for all $\alpha \neq 0$

* $A_{m \times n}$, we define the rank of A as $\operatorname{rank}(A)$

$= \dim(R(A))$

* $\operatorname{rank}(A) \leq \min\{m, n\}$

② $C(A) = \text{span}(a_1, a_2, \dots, a_n)$
Subspace of $\mathbb{R}^m \rightarrow \dim(\mathbb{R}^n) = m$
n columns
$\dim(C(A)) \leq n$

and $\therefore \dim(C(A)) \leq m$

$\dim C(A) \leq$

* If A, B are row equivalent
$C(A) \neq C(B)$

* If U is the R.R.E.F of A
Then
the columns of A have
the same relations as
columns of U

* Columns that have leading
ones in U form a basis
for C(U)

* Corresponding columns in A form a basis for C(A)

Ex: $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, find a basis and dim of R(A), C(A), N(A)

$$\rightsquigarrow \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$x_2 = \alpha$, $x_4 = \beta$

1. $\underline{N(A)} =$

$x_3 = 0$

$x_1 = \alpha - \beta$

$$N(A) = \left\{ \begin{pmatrix} \alpha-\beta \\ \alpha \\ 0 \\ \beta \end{pmatrix} : \alpha, \beta \text{ scalars} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \alpha, \beta \text{ scalar} \right.$$

basis for $N(A)$ is :-

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim(N(A)) = 2$

② :- $R(A)$ basis for $R(A)$ is

$$\left\{ \begin{pmatrix} 1 & -1 & 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

$\text{rank } R(A) = 2$

نبحث عن ~~العمود~~ الأعمدة التي تحتوي على l. ones

في U ثم بأخذ ترتيبهم

③ $C(A)$:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

هنا العمود الأول و الثالث من A

$\dim(C(A)) = 2$

def :- $\dim(N(A))$
$= \text{nullity}(A)$

Results

$A_{m \times n}$

1) $\dim N(A)$ = number of free variables

2) $\dim(R(A))$ = number of non zero rows in U
= number of leading ones
= " " " variables

3) $\dim(C(A))$ = number of leading ones.

So A Result :-

$\dim(R(A)) = \dim(C(A))$
$= \text{Rank}(A)$

Th: let $A_{m \times n}$ matrix Then
$\text{Rank}(A) + \text{nullity}(A) = n$
number of columns

Ex:- $\underset{5\times3}{A} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$

$0 \leq \text{rank}(A) \leq \min\{m, n\}$

$0 = \text{rank}(A) \iff A = 0_{m\times n}$

$R(A) = \{0\}$

has no basis

S.P. of $\mathbb{R}^4$

Ex: find a basis and dimensions of

$\text{Span}\left(x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 0 \end{pmatrix}, x_4 = \begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix} \right\}$

let $X = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{pmatrix}$

• Knowing that $C(A) = \text{Span}(a_1 - a_n)$

$S = C(A)$

$$\to U = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis for S are $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix} \right\}$

• $\mathbb{R}^{1\times n}$ تنتمي على

• $\mathbb{R}^{m\times 1} \to = \mathbb{R}^{m}$

Remark:- $R(A^T_{m\times n}) = C(A)$

* But we don't use it
Because we can find
C(A) using leading ones in
U

Th: let A be $m \times n$ - matrix
The system $Ax = b$ is consistent iff b can be written as a l.c of columns of A

iff $b \in C(A)$

Th: let A be a $m \times n$ - matrix

1. The linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ if and only if columns of A form a spanning set of $\mathbb{R}^m$

span for $C(A)$

2. The system $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$ if and only if columns of A are linearly independent

* special case $m = n$
$A_{n \times n}$ is nonsingular iff columns of A form a basis for $\mathbb{R}^n$

If $Ax=b$ is Consistent for every $b \in \mathbb{R}^m \Rightarrow$ every $b \in \mathbb{R}^m$ is in $C(A)$

$$\mathbb{R}^m = C(A)$$