lec 25:-

$\underbrace{R(A), C(A), N(A)}_{U}$

$\text{rank}(A) = \dim(R(A)) = \dim(C(A))$

* $\underset{m\times n}{A}x = b$ has at least one solution (is Consistent) for every $b \in \mathbb{R}^m$
$\iff$ Columns of A form a sp. set for $\mathbb{R}^m$

* $\underset{m\times n}{A}x = b$ has at most one solution for every $b \in \mathbb{R}^m \iff$ Columns of A are independent in $\mathbb{R}^m$

* $A_{m\times n}$
n: columns $\in \mathbb{R}^m$
Columns form a sp. set for $\mathbb{R}^m$

→ Since that:- $n \geq m$

→ If columns of A are L.I

$\Rightarrow n \leq m$

* If colums of A form a basis for $\mathbb{R}^m \Rightarrow n = m$

Theory:- $A_{n \times n}$ - matrix is nonsingular iff columns of A are L.I

$\Leftrightarrow \dim(C(A)) = n$ — span of $C(A)$ and l.I so it's a basis of $C(A)$ so

$\Leftrightarrow \text{rank}(A) = n$

$\dim(R(A))$ or $\dim(C(A))$

$\Leftrightarrow$ nullity $(A) = 0$

$\Leftrightarrow$ rows of A are L.I

columns of A form a basis for $\mathbb{R}^n$

$\Leftrightarrow$ " " " sp. set for $\mathbb{R}^n$

$\Longleftrightarrow$: rows form a sp. set for $\mathbb{R}^{1\times n}$

$\Longleftrightarrow$: rows form a basis for $\mathbb{R}^{1\times n}$

$\Longleftrightarrow$: $N(A) = \{0\}$

$\Longleftrightarrow$: $Ax = b$ has only ~~zero~~ unique solution for every $b \in \mathbb{R}^n$

if $b = 0 \rightsquigarrow x = 0$ (zero solution)

Question (13)

$A_{4\times 3}$ $\quad z_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $z_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

form a basis for $N(A)$

if $b = a_1 + 2a_2 + a_3$

$\hookrightarrow$ find all solutions to $Ax = b$

$x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a solution

to $Ax = b$

- Any solution to $Ax=b$ can be written as $y = \underbrace{x_0}_{\text{all solutions to } Ax=b} + z$, $z \in N(A)$ → homogeneous system

$$y = x_0 + z$$
$$= \begin{pmatrix}1\\2\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$N(A) = \left\{\alpha\begin{pmatrix}1\\1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$$

$$= \begin{pmatrix}1+\alpha+\beta\\ 2+\alpha\\ 1+2\alpha-\beta\end{pmatrix}$$

Question (15)

$A_{4\times5}$, $a_1 = \begin{pmatrix}2\\1\\-3\\-2\end{pmatrix}$, $a_2 = \begin{pmatrix}-1\\2\\3\\1\end{pmatrix}$

$$U = \begin{pmatrix}1 & 0 & 2 & 0 & -1\\ 0 & 1 & 3 & 0 & -2\\ 0 & 0 & 0 & 1 & 5\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

a) $N(A)$

b) Given $x_0 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ is a Sol to

$Ax = b$, $b = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \end{pmatrix}$

a) – find all sol to $Ax = b$

b) – find A

a) $x_3 = \alpha$, $x_5 = \beta$

$x_4 = -5\beta$, $x_2 = -3\alpha$

$x_1 = -2\alpha + \beta$

$\alpha, \beta \in \mathbb{R}$

$$N(A) = \left\{ \begin{pmatrix} -2\alpha + \beta \\ -3\alpha \\ \alpha \\ -5\beta \\ \beta \end{pmatrix} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{pmatrix} ; \alpha, \beta \in R \right\}$$

basis for $N(A)$ is $\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$

nullity $(A) = 2$

rank$(A) = 5 - 2 = 3$

rows $= 4 \geq$ rank $\Rightarrow$ L.D

all solutions of $Ax = b$ are

$$y = x_0 + z = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2\alpha + \beta \\ -3\alpha + 2\beta \\ \alpha \\ -5\beta \\ \beta \end{pmatrix}$$

$b = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \end{pmatrix}$, $x_0 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ is a sol to

$Ax = b \Longrightarrow b = 3a_1 + 2a_2 + 0a_3 + 2a_4 + 0a_5$

$$a_4 = \frac{\left(b - 3a_1 - 2a_2\right)}{2}$$

$$= \frac{\begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 1 \\ -3 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ -9 \\ -6 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 6 \\ 2 \end{pmatrix}$$

$$a_4 = \begin{pmatrix} -4 \\ -2 \\ -6 \\ 8 \end{pmatrix}$$

if α 1 $\beta = 0$

$X = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ is a sol to $Ax = b$

$b = (1)a_1 + (-1)a_2 + a_3 + 2a_4 + 0.a_5$

Then choose any solution and find as

$\alpha_1 = 0$, $\beta = 1$ for example

$$X = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -3 \end{pmatrix}$$ is a Sol for $Ax = b$

Back to $U = \begin{pmatrix} 1 & 0 & \text{---} \\ & \text{---} & \\ & \text{---} & \end{pmatrix}$

$u_3 = 2u_1 + u_2$

$a_3 = 2a_1 + 3a_2 = \begin{pmatrix} 1 \\ 8 \\ \\ -1 \end{pmatrix}$

Chapter 3 is finished

Chapter 4:- linear Transformations

Def:- A function $L: V \to W$

where V, W are vector spaces
is called a linear Combination
if

$$L(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 L(V_1) + \alpha_2 L(V_2)$$

$V_1, V_2 \in V$, $\alpha_1, \alpha_2 \in \mathbb{R}$

Ex:- IS $L: \mathbb{R}^3 \to \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

is L. T = ?

$$L\left(\alpha_1\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\alpha_2\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right)$$

$$=L\left(\begin{pmatrix}\alpha_1x_1\\\alpha_1x_2\\\alpha_1x_3\end{pmatrix}+\begin{pmatrix}\alpha_2y_1\\\alpha_2y_2\\\alpha_2y_3\end{pmatrix}\right)$$

$$=L\begin{pmatrix}\alpha_1x_1+\alpha_2y_1+\alpha_1x_2+\alpha_1y_2\\\alpha_1x_2+\alpha_2y_2+\alpha_1x_3+\alpha_2y_3\end{pmatrix}$$

$$=\begin{pmatrix}\alpha_1x_1+\alpha_1x_2\\\alpha_1x_2+\alpha_1x_3\end{pmatrix}+\begin{pmatrix}\alpha_2y_1+\alpha_2y_2\\\alpha_2y_2+\alpha_2y_3\end{pmatrix}$$

to $Ax=b$

$$=\alpha_1\begin{pmatrix}x_1+x_2\\x_2+x_3\end{pmatrix}+\alpha_2\begin{pmatrix}y_1+y_2\\y_2+y_3\end{pmatrix}$$

$$=\alpha_1L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\alpha_2L\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

So L is L.T

Ex, $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2 \\ x_1 + x_2 \end{pmatrix}$$

Is L L.T ?

$$L\left(\alpha_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) =$$

$$= L\begin{pmatrix} \alpha_1 x_1 + \alpha_2 y_1 \\ \alpha_1 x_2 + \alpha_2 y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 x_1 + \alpha_2 y_1 + 2 \\ \alpha_1 x_1 + \alpha_2 y + \alpha_1 x_2 + \alpha_2 y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 x_1 + 2 \\ \alpha_1 x_1 + \alpha_1 x_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} y_1 \\ y_1 + y_2 \end{pmatrix}$$

is not a L.T

If $L: V \to V$ linear operator

Ex: $L: P_3 \longrightarrow \mathbb{R}^2$

$$L(P(x)) = \begin{pmatrix} \int_0^1 P(x)\,dx \\ P'(0) \end{pmatrix}$$

$$L(3x^2+x+1) = \begin{pmatrix} \int_0^1 (3x^2+x+1)\,dx \\ (6x+1) \end{pmatrix}_{x=0}$$

$$= \begin{pmatrix} 1 + \frac{1}{2} + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$$

L.T ?

$$L(\alpha_1 p(x) + \alpha_2 q(x)) =$$

$$= \begin{pmatrix} \int_0^1 \alpha_1 p(x) + \alpha_2 q(x)\,dx \\ (\alpha_1 p(x) + \alpha_2 q(x))'(0) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 \int_0^1 p(x)\,dx + \alpha_2 \int_0^1 q(x)\,dx \\ \alpha_1 p'(0) + \alpha_2 q'(0) \end{pmatrix}$$

$$= \alpha_1 \begin{pmatrix} \int_0^1 p(x)\,dx \\ p'(0) \end{pmatrix} + \alpha_2 \begin{pmatrix} \int_0^1 q(x)\,dx \\ q'(0) \end{pmatrix}$$

$$= \alpha_1 L(p(x)) + \alpha_2 L q(x)$$