lec 26

$L: V \to W$

$LT \iff L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

1. $L(v_1 + v_2) = L(v_1) + L(v_2)$
2. $L(\alpha v) = \alpha L(v)$

Ex: $L: \mathbb{R}^3 \longrightarrow P_3$

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bx + cx^2$$

Is L a $L.T$?

$$L\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right) = L\begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}$$

$$= a_1 + a_2 + (b_1 + b_2)x + (c_1 + c_2)x$$

$$= (a_1 + b_1 x + c_1 x^2) + (a_2 + b_2 x + c_2 x^2)$$

$$= L\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + L\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

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② $L\left(\alpha\begin{pmatrix}a\\b\\c\end{pmatrix}\right) = L\begin{pmatrix}\alpha a\\ \alpha b\\ \alpha c\end{pmatrix}$

$= \alpha a + (\alpha b)x + (\alpha c)x^2$

$= \alpha(a + bx + cx^2)$

$= \alpha L\begin{pmatrix}a\\b\\c\end{pmatrix}$

so L is a L.T

Proprieties :-

if $L: V \to W$ is a L.T

① $L(0_V) = 0_W$

$L(0_V) = L(0 \cdot 0_V)$

$= 0 \cdot \underline{L(0_V)}$ ↳ vector in W

$= 0_W$

if image of zero equals 0

That doesn't have to mean

That the function is L.T

But if it's not it's not L.T ~~Then It Must~~

②: $L(\alpha_1 V_1 + (\alpha_2 V_2 + \cdots + \alpha_n V_n)$
$= L(\alpha_1 V_1) + L(\alpha_2 V_2)$
$+ (\cdots + \alpha_n V_n$
$= \alpha_1 (L)(V_1) + \alpha_2 L(V_2) + \alpha_3 L(V_3)$
$+ \cdots + \alpha_n L(V_n)$

③ $L(-V) = -L(V)$

$L(-V) = L((-1)(V))$
$= (-1) L(V)$
$= -L(V)$

Ex: If $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a L.T

and $L\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

$L\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$

$L\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}7\\8\end{pmatrix}$

basis

a) find $L\begin{pmatrix}3\\4\\5\end{pmatrix}$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$ is a basis for $\mathbb{R}^3$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & 0 & -1 & 4 \\ 1 & 1 & 0 & 5 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

So $\alpha_1 = 3+1 = 4$ / $\alpha_2 = 1$ / $\alpha_3 = 0$

$$L\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = L\left(4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$$

$$= 4\,L\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\cdot L\begin{pmatrix}-1\\0\\1\end{pmatrix} + 0\cdot L\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= 4\begin{pmatrix}2\\3\end{pmatrix} + 1\begin{pmatrix}5\\0\end{pmatrix} + 0\begin{pmatrix}7\\8\end{pmatrix}$$

$$= \begin{pmatrix}13\\18\end{pmatrix} = L\begin{pmatrix}3\\4\\5\end{pmatrix}$$

Remark:-

standard Basis

$$L\begin{pmatrix}3\\4\\5\end{pmatrix} = L\left(3\begin{pmatrix}1\\0\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

b) $L\begin{pmatrix}a\\b\\c\end{pmatrix} = ?$ find a formular for L

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}-1\\0\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 0 & -1 & b \\ 1 & 1 & 0 & c \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 2 & -1 & c-a \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 3 & c+a-2b \end{array}\right)$$

$\alpha_1 = a + b - a + \frac{2(c+a-2b)}{3} - \frac{c+c}{3}$

$\alpha_2 = b - a + 2\left(\frac{c+a-2b}{3}\right)$

$\alpha_3 = \frac{c+a-2b}{3}$

$$\alpha_1 = b + \frac{c+a-2b}{3} = \frac{1}{3}(b+c+a)$$

$$\alpha_2 = -\frac{1}{3}b - \frac{1}{3}a = \frac{1}{3}(2c-b-a)$$

$$\alpha_3 = \frac{c+a-2b}{3}$$

$$L\begin{pmatrix}a\\b\\c\end{pmatrix} = L\left(\frac{1}{3}(b+c+a)\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}(2c-b-a)\begin{pmatrix}-1\\0\\1\end{pmatrix} + \frac{1}{3}(c+a-2b)\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)$$

$$= \frac{1}{3}(b+c+a)\,L\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}(2c-a-b)\,L\begin{pmatrix}-1\\0\\1\end{pmatrix} + \frac{1}{3}(c+a-2b)\,L\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= \frac{1}{3}(b+c+a)\begin{pmatrix}2\\3\end{pmatrix} + \frac{1}{3}(2c-a-b)\begin{pmatrix}5\\6\end{pmatrix} + \frac{1}{3}(c+a-2b)\begin{pmatrix}7\\8\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{3}(b+c+a) + \frac{5}{3}(2c-b-a) + \frac{7}{3}(c+a-2b) \\ b+c+a + 2(2c-b-a) + \frac{8}{3}(c+a-2b)\end{pmatrix}$$

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* $A_{m\times n}$ matrix

$$L_A : \mathbb{R}^n \to \mathbb{R}^m$$

$$L(\underset{n\times 1}{X}) = \underset{m\times 1}{\left(\underset{m\times n}{A}\underset{n\times 1}{X}\right)}$$

Is L a L.T?

- $L(\alpha_1 x + \alpha_2 y) = A(\alpha_1 x + \alpha_2 y)$

$$= \alpha_1 A(x) + \alpha_2 A(y)$$

$$= \alpha_1 L(x) + \alpha_2 L(y)$$

$\rightsquigarrow$ L_A is a L.T

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T

is There a matrix A s.t

$$T(x) = Ax$$

$$L: \mathbb{R}^3 \to \mathbb{R}^2$$

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+x_2\\x_3+x_2\end{pmatrix}$$

$$?\ A_{2\times3} : L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\end{pmatrix}_{2\times3}$$

$$L(x) = A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$= \begin{pmatrix}x_1+x_2\\x_2+x_3\end{pmatrix}$$

Def: let $L: V \longrightarrow W$ be a L.T. Kernel of L is defined as

$$\text{Ker}(L) = \{v \in V : L(v) = 0\}$$

roots of L

① $0_V \in Ker(L)$ since $L(0) = 0$

$v \in Ker(L) \iff L(v) = 0$

Ex: $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

$$Ker(L) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$x_1 + x_2 = 0$
$x_2 + x_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \therefore \begin{array}{l} x_3 = \alpha \text{ free} \\ x_2 = -\alpha \\ x_1 = \alpha \end{array}$$

$\cancel{\begin{pmatrix} 1 & 1 & & & \end{pmatrix}}$

$$Ker(L) = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} , \alpha \in \mathbb{R} \right.$$

Theory :-

Let $L : V \longrightarrow W$ be a L.T
the $Ker(L)$ is a subspace of V

Proof :- $Ker(L) = \{ v \in V : L(v) = 0 \}$

① $Ker(L) \neq \phi$, $0 \in Ker(L)$

② let $V_1, V_2 \in Ker(L)$

$\Longrightarrow L(V_1) = 0$, $L(V_2) = 0$

$V_1 + V_2 \in Ker(L)$??

$$L(v_1+v_2) = L(v_1) + L(v_2) = 0+0$$

$$v_1+v_2 \in Ker(L)$$

③ $v \in Ker(V)$, α scalar

$\Longrightarrow L(v) = 0$

$$L(\alpha v) = \alpha L(v) = \alpha(0) = 0$$

$$\alpha v \in Ker(L)$$

Ker(L) is a subspace of V

- Def: let $L: V \to W$ be a L.T
Image of L (Range of L) is defined as $Im(L) = \{L(v) : v \in V\}$

$$Im(L) = \{w \in W : \text{There exists } v \in V \text{ with } L(v) = w\}$$

① Im(L) is a subspace of W

① $Im(L) \neq \phi$ since $V \neq \phi$ $(0 \in V)$

$$0 = L(0) = Im(L)$$

② Let $w_1, w_2 \in Im(L)$

$\exists\, V_1, V_2 \in V$

$L(V_1) = W_1$, $L(V_2) = W_2$

$W_1 + W_2 \overset{?}{\in} Im(L)$??

$W_1 + W_2 = L(V_1) + L(V_2)$

$= L(V_1 + V_2)$, $V_1 + V_2 \in V$

So $W_1 + W_2 \in Im(L)$

③ ✓

Remark: $Im(L) = L(V)$

$\rightsquigarrow$ If S subspace of V, Then $L(S)$ is a subspace of W

Ex: find a basis of and dimension of $Ker(L)$ and $Im(L)$, where $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

① $Ker(L) = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$

basis for $Ker(L)$ is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\dim(Ker(L)) = 1$

② $Im(L) = \left\{ L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$

$= \left\{ \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}$

$= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right.$

$\left. x_1, x_2, x_3 \in \mathbb{R} \right\}$

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Sp set for $Im(L)$ is

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

L.D $\longrightarrow$ remove one of Them

remove $\begin{pmatrix}1\\1\end{pmatrix}$

$\rightsquigarrow$ $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ is a sp. set for $Im(L)$

and since They are The stan Basis for $Im(L)$

is $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$\dim(Im(L)) = 2$

$\therefore Im(L) = \mathbb{R}^2$

L is onto

L is 1-1 iff

$Ker(L) = \{0\}$