lec 28:-

Chapter 6:

• 6.1: Eigenvalues of a matrix

Ex:- $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$Ax = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \underline{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

matrix Multiplication

scalar Multiplication

$$Ax = \underline{\lambda} x$$ ← Eigenvector

Eigenvalue

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

is there λ , $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ s.t. $Ax = \lambda x$

Def:- Scalar ≡ Complex number

let A be $n \times n$ - matrix

A scalar λ is called

an eigenvalue for A iff
There exists a nonzero vector $x \in \mathbb{R}^n$
s.t. : $Ax = \lambda x$
This x (if exists is called an eigenvector corresponding to λ)

Ex:- $\lambda = 3$ is eigenvalue for $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$
with eigenvalue $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$y = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$Ay = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$y = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is eigenvector for $\lambda = 3$

let $z = \begin{pmatrix} 2a \\ a \end{pmatrix}$, a scalar $\neq 0$

$Az = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2a \\ a \end{pmatrix} = \begin{pmatrix} 6a \\ 3a \end{pmatrix}$

$\forall$ a scalar $= 3 \begin{pmatrix} 2a \\ a \end{pmatrix} = 3z$

* $A_{n\times n}$

- λ is eigenvalue for $A \Leftrightarrow \exists\, x \neq 0 \in \mathbb{R}^n$ s.t $Ax = \lambda x \Longleftrightarrow \exists\, 0 \neq x \in \mathbb{R}^n$ s.t $Ax - \lambda x = 0$

$\Longleftrightarrow 0 \neq x \in \mathbb{R}^n$ s.t. $(A - \lambda I)x = 0$

homogenuous system

$\Longleftrightarrow$ The homogenuous system $(A - \lambda I)x = 0$ has nonzero solution

$\Longleftrightarrow$ $(A - \lambda I)$ is singulare

$\det(A - \lambda I) = 0$

- eigenvalues for A are the roots (solutions) of $[\det(A - \lambda I) = 0]$

- eigenvalues are the nonzero solutions to the system $(A - \lambda I)x = 0$

Ex:- $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, find the eigenvalues and the corresponding eigenvectors of A

Solve $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$2 + \left((4-\lambda)(1-\lambda) \right) = 0$$
$$2 + \left(4 - 4\lambda - \lambda + \lambda^2\right) = 0$$
$$2 + {}^{-}5\lambda + \lambda^2 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-3)(\lambda-2) = 0$$

2 eigen values $\lambda_1 = 2$, $\lambda_2 = 3$

Eigenvectors: ① for $\lambda_1 = 2$

solve

$$(A - 2I)(X) = 0$$

$$\begin{pmatrix} 4-2 & -2 \\ 1 & 1-2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0$$

$$\left(\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_2 = \alpha$
$x_1 = \alpha$

Eigenvectors for $\lambda_1 = 2$ are $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$

$\alpha \neq 0$

Eigenspace for $(\lambda_1 = 2)$ is

$$E(\lambda_1 = 2) = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} ; \alpha \text{ is a scalar} \right\}$$

$N(A - 2I)$

$E(\lambda)$ is a subspace of $\mathbb{R}^n$.

Basis for $E(\lambda_1 = 2)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\dim (E(\lambda_1 = 2)) = 1$

2) for $\lambda_2 = 3$

Eigenvectors $(A - 3I) = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$x_2 = \beta$
$x_1 = 2\beta$

$E(\lambda_2 = 3) = \left\{ \begin{pmatrix} 2\beta \\ \beta \end{pmatrix} : \beta \text{ scalar} \right\}$

Eigenvectors $\begin{pmatrix} 2\beta \\ \beta \end{pmatrix}, \beta \neq 0$

$= \left\{ \beta \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{Basis}} : \beta \text{ scalar} \right\}$

$\dim = 1$

- Eigenvectors does not form a Subspace Because $0 \notin$

But Eigenspace is a subspace

$$\Rightarrow E(\lambda_1=2) \cap E(\lambda_2=3) = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

* $A_{n\times n}$

$p(\lambda) = |A - \lambda I|$ • Characteristic polynomial of degree $\underline{n}$

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$(a_{11}-\lambda) \begin{vmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \\ a & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\underbrace{|\ \ |}_{(n-2)(n-2)} + \overbrace{\quad\quad}^{(n-1)(n-1)}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + \cdots$$

$$= (-1)^n \lambda^n$$

② roots of $P(\lambda)$ are the eigenvalues for A

(Solve $P(\lambda) = 0$)

Complex , $a + ib$ $a, b \in \mathbb{R}$

$P(\lambda) = \lambda^2 + 1$

↳ has two roots $\pm i$

* $P(\lambda) = (\lambda-1)^3 = (\lambda-1)(\lambda-1)(\lambda-1)$

3 roots 1, 1, 1

They are different

$P(\lambda) = (\lambda - 1)$ has 1 root $\lambda = 1$

* $\lambda = 1$ is a Multiple root (multiplicity = 3)

* $P(\lambda)$, $\deg P(\lambda) = n$

→ it has n roots in $\mathbb{C}$ complex numbers

(Counting multiplicity)

- distinct roots :- multiplicity is not Counted

* roots of $P(\lambda)$ are the eigenvalues of A

$\therefore A_{n \times n}$ has n eigenvalues (counting multiplicity

distinct eigenvalues $\leq n$

Ex. $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

find eigenvalues and eignspaces

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 0 & -1+\lambda & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((1-\lambda)(-2-\lambda) - (-1+\lambda)\big) - 1\big(-3(1-\lambda) - (\lambda-1)\big)$$

$$= (2-\lambda)(\lambda^2+\lambda-1) - (-2+2\lambda)$$

$$= 2\lambda^2 - 2 - \lambda^3 + \lambda + 2 - 2\lambda$$

$$= -\lambda^3 + 2\lambda^2 - \lambda$$

Solve $P(\lambda) = 0 \quad \lambda^3 - 2\lambda^2 + \lambda = 0$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1$$

Eigenvalues

① $\lambda_1 = 0$

$$A - 0I = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = \alpha$

$x_2 = \alpha$

$x_1 = 2\alpha - \alpha = \alpha$

$$E(\lambda = 0) = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalars} \right\}$$

Eigenvectors $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$, α scaler

(2) $\lambda_2 = \lambda_3 = 1$

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_2 = \beta \\ x_3 = \gamma \\ x_1 = 3\beta - \gamma \end{array}$$

So Eigonspace for $\lambda_2 = \lambda_3 = 1$

is $\left\{ \begin{pmatrix} 3\beta - \gamma \\ \beta \\ \gamma \end{pmatrix}, \beta, \gamma \text{ scalars} \right\}$

$= \left\{ \beta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ β, γ scalars

$\lambda = 0$ is eigonvectors of A

$\Longleftrightarrow P(0) = 0$

$|A - 0I| = 0$

$|A| = 0$ A is nonsinegular