lec29:-

* $\lambda=0$ is an eigenvalue for A
A is singular

$|A|=|A-0I|=0$

Ex :

$$A=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$P(\lambda)=|A-\lambda I|=\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)^2+4=\lambda^2-2\lambda+5$$

Solve $P(\lambda)=0 \iff \lambda^2-2\lambda+5=0$

$$\lambda_{1,2}=\frac{2\pm\sqrt{4-20}}{2}$$

$$=1\pm 2i$$

$\lambda_1=1+2i$, $\lambda_2=1-2i$ $(\bar{\lambda}_1=\lambda_2)$

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Eigenvectors ① $\lambda_1 = 1+2i$

$$A - \lambda_1 I = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}$$

$$iR_1 + R_2 \rightsquigarrow \begin{pmatrix} -2i & 2 \\ 0 & 0 \end{pmatrix}$$

$X_2 = \alpha$ free

$-2i\, x_1 = -2\alpha$

$x_1 = -i\alpha$

$$E(\lambda_1 = 1+2i) = \left\{ \begin{pmatrix} -i\alpha \\ \alpha \end{pmatrix} : \alpha \text{ scalar} \right\}$$

$X = \begin{pmatrix} 1 \\ i \end{pmatrix}$ eigenvector for λ_1

if $\alpha = i$ $\quad X = \begin{pmatrix} 1 \\ i \end{pmatrix}$

if $\alpha = 1+i$ $\quad X = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

basis for $E(\lambda_1)$

$$= \left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

② $\lambda_2 = 1-2i$

$$A - \lambda_2 I = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}$$

$$\xrightarrow{-iR_1+R_2} \begin{pmatrix} 2i & 2 \\ 0 & 0 \end{pmatrix}$$

$x_2 = \beta$ free

$2i\, x_1 = -2\beta$

$x_1 = i\beta$

$$E(\lambda_2) = \left\{ \begin{pmatrix} i\beta \\ \beta \end{pmatrix} : \beta \text{ scalar} \right\}$$

Basis $E(\lambda_2) = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$.

* Remarks :-

① If A is real matrix
$(a_{ij} \in \mathbb{R})$

- If $z = a + ib$ is an eigenvalue for A with eigenvector X

① $\bar{\lambda} = a - ib$ is also an eigenvalue

② ~~eigen vale~~ eigenvector for $\bar{\lambda}$ is $\bar{X}$

$X = \begin{pmatrix} 1 \\ i \end{pmatrix}$ eigenvector $\lambda = 1 + 2i$

$\bar{X} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ ~ ~ $\bar{\lambda} = 1 - 2i$

② $A_{n \times n}$ with eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may be repeated)

$\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$

$P(\lambda) = |A - \lambda I|$

roots are eigenvalues

Remark. -

$P(x)$, $\deg(P) = 3$

roots are $1, 1, 2$

$P(x) = (x-1)(x-1)(x-2)$

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$P(0) = \lambda_1 \lambda_2 \cdots \lambda_n = |A|$ دايماً موجبة

② $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

$$= \sum_{i=1}^{n} a_{ii} = \text{Trace of } A$$

Sum of elements of A on the main diagonal

Ex: $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$\lambda_1 = 1+2i \quad , \quad \lambda_2 = 1-2i$

1) $\lambda_1 \lambda_2 = (1+2i)(1-2i)$
$= 5 = \det(A)$

2) $\lambda_1 + \lambda_2 = 2 = \text{trace}(A)$

Def:- (similar matrices)

A matrix $B_{n\times n}$ is called similar to a matrix $A_{n\times n}$ if ~~there~~ exists a nonsingular matrix S s.t $A = S^{-1} B S$

If B similar to A, Then A is similar to B

Proof:- $\Rightarrow \exists S$ nonsingular s.t
$A = S^{-1} B S$

$$\Longrightarrow \quad SAS^{-1} = B, \quad \boxed{\overset{\text{let}}{S^{-1} = D}}$$

$$B = D^{-1} A D$$

- Theory: if A, B are similar matrices Then They have the same characteristic Polynomial (and So The same eigenvalue)

Proof:- Assume A, B are $n \times n$-Similar matrices $\Longrightarrow \exists S$ nonsingular matrix s.t $B = S^{-1} A S$

$$P_B(\lambda) = |B - \lambda I|$$

$$= |S^{-1} A S - \lambda I|$$

$$= |S^{-1} A S - S^{-1} \lambda I S|$$

$$= |S^{-1} (AS - \lambda S)|$$

$$= |S^{-1} (A - \lambda) S|$$

$$= |S^{-1}| \, |A - \lambda| \, |S|$$

$$|A-\lambda I| = P_A(\lambda)$$

G.3:-

A^2

$A^{37}_{5\times 5}$

1) If $D_{n\times n}$ is diagonal matrix

$$D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & d_n \end{pmatrix}_{n\times n}$$

$$D^2 = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & \\ & & & \\ & & & d_n \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_n \end{pmatrix}$$

$$= \begin{pmatrix} d_1^2 & 0 & \cdots & \\ 0 & d_2^2 & \cdots & \\ 0 & \cdots & d_3^2 & \cdots \\ & & & d_n^2 \end{pmatrix}$$

$$D^k = \begin{pmatrix} d_1^k & 0 & \cdots & \cdots & 0 \\ 0 & d_2^k & \cdots & \cdots & 0 \\ 0 & 0 & d_3^k & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_n^k \end{pmatrix}$$

2) If A is similar to $D \Rightarrow A = S^{-1} D S$

$k = 2$

$$A^k = A^2 = S^{-1} D S S^{-1} D S$$
$$= S^{-1} D^2 S$$

$$A^{10} = S^{-1} D \underline{S S^{-1}} D \underline{S S^{-1}} D \underline{S} \cdots \underline{S^{-1}} D S$$

$$= S^{-1} D^{10} S$$

$$A^k = S^{-1} D^k S$$

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$$

is A similar to diagonal matrix

Def: A matrix A is called

diagonalizable if A is similar to a diagonal matrix

$\left\{ \Longleftrightarrow \exists X_{n\times n}\right.$ nonsingular matrix

$D_{n\times n}$ diagonal matrix

s.t $X^{-1}AX = D \Big\}$

$A = XDX^{-1}$

Theory: let A be $n\times n$ matrix if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues for A with eigen Values $X_1, X_2, \ldots X_k$

Then $X_1, X_2, \ldots X_k$ are linearly independent

Ex:- $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

$\lambda_1 = 0 \rightsquigarrow E(\lambda_1 = 0) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right.$

α scaler $\Big\}$

$\lambda_2 = \lambda_3 = 1$

$$E(\lambda_2 = \lambda_3 = 1) = \left\{ \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

α, β scalars

- max # of L.I eigenvectors

= Sum of dimensions of all eigenspaces

In This example:-

= 3

6.3 Diagonalization

Def:- A matrix is called diagonalizable if A is similar to a diagonal matrix

* A is diagonalizable $\Longleftrightarrow \exists$ a non singuler matrix X and a diagonal matrix D s.t

$$X^{-1}AX = D \quad (A = XDX^{-1})$$

* if $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues for A with eigenvectors $x_1, x_2, \dots, x_k$

Then $x_1, \dots, x_k$ are L.I

* Theory - A matrix $A_{n\times n}$ ~~of~~ is diagonalizable iff
A has n linearly independent eigenvectors and

~~D~~:

X: Columns of X are the n L.I eigenvectors

D: diagonal elements of D are the corresponding eigenvalues (same order as in X)

$$X = \begin{pmatrix} \begin{pmatrix} \\ x_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ x_3 \\ \\ \end{pmatrix} \cdots x_n \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

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Ex: $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

Is A diagonalizable?

If yes find $\underbrace{X}$, D.

diagonalizing matrix for A

$P(\lambda) = -\lambda(\lambda-1)^2$

$\lambda_1 = 0 \quad \therefore \lambda_2 = \lambda_3 = 1$

① :- $E(\lambda=0) = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}, \ \alpha \text{ scaler} \right\}$

basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

② $E(\lambda_2=\lambda_3=1) = \left\{ \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; \alpha, \beta \text{ Scalar} \right\}$

basis for $E(\lambda_2=\lambda_3=1)$ is

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- Total number of L.I eigenvectors $= 3$

So A is diagoalizable

$$X = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}_{3\times3}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

L.I eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

- You can choose any order of these three and D depends on this order. So D is defined By X

Remarks :-

① If $A_{n\times n}$ has n distinct eigenvalues Then A is diagonalizable

If $A_{n\times n}$ has less than n distinct eigenvalue Then A may or may not be diagonalizable

② X, D are not unique

Ex:- $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}$

• If A is lower or upper triangular, eigenvalues are the elements on the diagonal

Starting with A :-

① $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$$P_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda)(2-\lambda) = (2-\lambda)^2(4-\lambda)$$

So Eigenvalues $P(\lambda) = 0$

$\lambda_2 = \lambda_3 = 2, \quad \lambda_1 = 4$ 　　 $\dim(E) \leq$ multiplicity

$$A - 4I = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 =$ free $= \alpha$

$x_3 = 0$

$x_1 = 0$

$$E(\lambda_1 = 1) = \left\{ \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} ; \alpha \text{ scaler} \right\}$$

$$\text{basis} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

② $E(\lambda_2 = \lambda_3 = 2)$

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} x_3 = \beta \\ x_2 = 0 \\ x_1 = 0 \end{array}$$

$$E(\lambda_2 = \lambda_3 = 2) = \left\{ \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} ; \beta \text{ scaler} \right\}$$

$$\text{basis} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

So A has 2. L.I . eigenvectors
So A is not diagonalizable
(Defective)

B $E(\lambda_1 = 4)$

$$B - 4I = \begin{pmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ -3 & 6 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 0 \\ -3 & 6 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E(\lambda_1 = 4) = \left\{ \begin{pmatrix} 0 \\ \alpha/3 \\ \alpha \end{pmatrix} \right\}$$

$X_3 = \alpha$ free

$X_2 = \alpha/3$

$X_1 = 0$

basis $= \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \end{pmatrix}$

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② $E(\lambda_2 = \lambda_3 = 2)$

$$B - 2I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 2 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = \beta, \quad x_3 = \gamma$

$x_1 = 2\beta$

$$E(\lambda_2 = \lambda_3 = 2) = \left\{ \begin{pmatrix} 2\beta \\ \beta \\ \gamma \end{pmatrix} : \beta, \gamma \text{ sca} \right\}$$

$$= \left\{ \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Basis

B has 3 L.I eigenvalues

So B is diagonalizable

$$X = \begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\bar{X} = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 5 \end{pmatrix}$$

$$\bar{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$