lec 4 :-

• Types of linear systems :-

1) Underdetermined system if $m < n$

Ex:- 1- $\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \end{aligned}\right\}$ 2×3 system

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \end{array}\right)$$

① $-2R_1 + R_2$

② $\frac{-1}{5} R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array}\right)$$

Consistent $\qquad x_3 = \alpha$ free

$x_2 = -\frac{1}{5}\alpha$

$x_1 = 1 + 2\left(\frac{1}{5}\alpha\right) - \alpha = 1 - \frac{3}{5}\alpha$

The system has infinit number of solutions

$$X = \begin{pmatrix} 1 - \frac{3}{5}\alpha \\ -\frac{1}{5}\alpha \\ \alpha \end{pmatrix}$$

Ex 2 - $4x_1 + 3x_2 + 3x_3 = 4$

$2x_1 - x_2 + 3x_3 = 5$

$$\left(\begin{array}{ccc|c} 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 4 & 3 & 3 & 4 \\ 0 & -\frac{5}{2} & \frac{3}{2} & 3 \end{array}\right) \quad ① \; -\tfrac{1}{2}R_1 + R_2$$

Cons $\quad x_3 = \alpha$ free $\rightarrow$ inf # of variable

Ex 3 - $x_1 + 2x_2 + x_3 = 1$

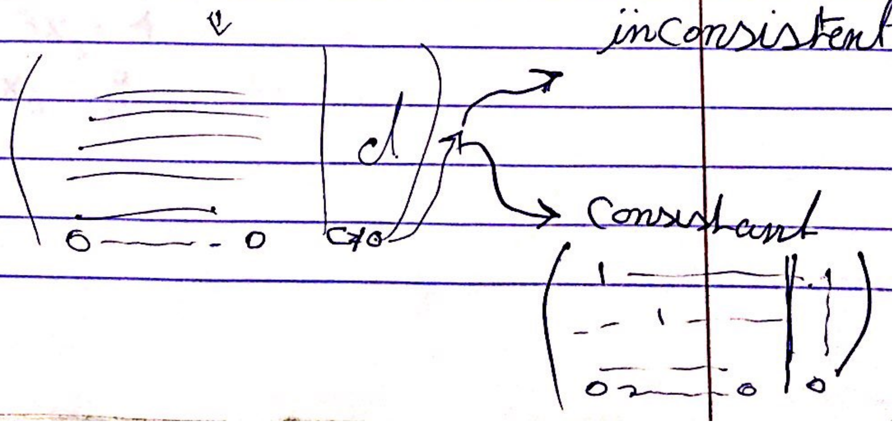
$2x_1 + 4x_2 + 2x_3 = 3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \quad ① \; -2R_1 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \text{inconsistant system}$$

$m \times n$ - system , $m < n$:-

$(A|b) \longrightarrow (U|d)$



$m-1 < m < n$

Explanation: # of leading ones in $(U|d) \leq$ # of rows

يعني أقصى عدد لـ ones يا بيساوي الـ rows يا أقل .

# of leading variables $\leq m$

Since we know that $m < n$

$$\underbrace{n - \text{\# of leading variables}}_{\text{number of free variables}} > 0$$

$\Longrightarrow$ So there are free variables

$\Longrightarrow$ The system has infinit number of solutions

- Undeterment system can't have one solution only

Theory :- A consistant & underdetermined system has infinit number of solutions

2) overdetermined system :

$m > n$

Ex: $x_1 + x_2 = 1$
$x_1 - x_2 = 3$
$-x_1 + 2x_2 = -2$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right)$$

① $-R_1 + R_2$
② $R_1 + R_3$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{array}\right)$$

③ $-3R_2 + R_3$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & \div 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

inconsistant system

Ex :- $x_1 + 2x_2 + x_3 = 1$
$2x_1 - x_2 + x_3 = 2$
$4x_1 + 3x_2 + 3x_3 = 4$
$3x_1 + x_2 + 2x_3 = 3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Cons

$X_3 = \alpha$ free

$X_2 = -\frac{1}{5}\alpha$

$X_1 = 1 - \frac{3}{5}\alpha$

Ex:- $X_1 + X_2 + X_3 = 1$

$2X_1 + -X_2 + X_3 = 2$

$4X_1 + 3X_2 + 3X_3 = 4$

$2X_1 - X_2 + 3X_3 = 5$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right)$$

① $-2R_1 + R_2$
② $-4R_1 + R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \text{only one solution.}$$

Cons
3 leading var
No free var

$X_1 = 1 + 2\times\frac{3}{10} - \frac{3}{2} = 1 - \frac{9}{10} = \frac{1}{10}$

$X_2 = -\frac{3}{10}$

$X_3 = \frac{3}{2}$

$$X = \begin{pmatrix} \frac{1}{10} \\ -\frac{3}{10} \\ \frac{3}{2} \end{pmatrix}$$

③ square system $(m = n)$

No sol — one sol — inf # of sol

always consist

4) Homogenuous system ($m \times n$ - system)

if $b_1 = b_2 = b_3 = \cdots = b_m = 0$

Ex: $\left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right)$

$-X_1 + X_2 - X_3 + 3X_4 = 0$
$3X_1 + X_2 - X_3 - X_4 = 0$
$2X_1 - X_2 - 2X_3 - X_4 = 0$

Uncle & homo

$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right)$

$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right)$

$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$

$X_3 = \alpha \qquad X_4 = \alpha$

$X_2 = X_3 - 2X_4$

$= \alpha - 2\alpha = -\alpha$

$X_1 = X_2 - X_3 + 3X_4 = -\alpha - \alpha + 3\alpha = \alpha$

$$X = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \ \alpha \in \mathbb{R}$$

Result :- ① Any homogeneous $m \times n$ system is consistent

has at least the zero solution

$$X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{Trivial solution}$$

Ex:- ②

$$\left.\begin{aligned} X_1 + X_2 - X_3 &= 0 \\ X_1 - X_2 + 2X_3 &= 0 \\ -X_1 + X_2 - 3X_3 &= 0 \end{aligned}\right\} \text{Cons}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

No free variable $\longrightarrow$ one solution.

There is only a zero solution $X = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Theory :- Any homogenuous and ~~det~~ underdetermany system has infinite number of solutions (has non zero solutions)

واحد من الحلول ليس صفر