lec 7:

Ex:- $x_1 + 2x_2 + 3x_3 = 2$
$-x_1 - x_2 + 2x_3 = -3$
$2x_1 - 3x_2 - x_3 = 1$

3 ways to write a system :-

① $(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ -1 & -1 & 2 & -3 \\ 2 & -3 & -1 & 1 \end{array}\right)$

② $Ax = b$ $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

③ $b = x_1 a_1 + x_2 a_2 + x_3 a_3$

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

* if $a_1, a_2, a_3, \dots, a_n$ are vectors (column matrix) a sum of the form $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$ is called a linear combination of $a_1, a_2, \dots, a_n$ {α is a scalar}

Ex:- Can we write $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solve :- $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

• If There is a one Solution → Yes we can write it as a linear Combination

" " " more Than one Solution → Yes we Can " " " But in different Ways

• If not (No Sol found) The answer is No

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & -1 & 4 \end{array}\right)$$

⇓

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -7 & -2 \end{array}\right)$$

⇓

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & \frac{2}{7} \end{array}\right)$$ No free variables

$x_2 = \frac{2}{7}$

$x_1 + \frac{4}{7} = 2 \longrightarrow x_1 = \frac{10}{7}$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{10}{7}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ can be written as a linear combination of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ - \end{pmatrix}$

Th:- The system is $Ax = b$ is Consistant if and only if b can be written as a linear combination of columns of A

* $Ax = b \Rightarrow b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$

* non-singular matrix:-

If $AB = I = BA$, $A_{n \times n}$, $B_{n \times n}$

↳ A, B are nonsingular, each is the inverse of the other

* If $A_{n \times n}$ is nonsingular $\Rightarrow$ A has an inverse B

Then B is the only matrix that satisfy $AB = I = BA$ B is unique

• To Prove B is unique :-
Assume B, C are both inverses of A
$\Rightarrow AB = I = BA$ and $AC = \underbrace{I}{} = \underbrace{CA}{}$
Show $B = C$

$$B = IB = (CA)B = C(AB) = CI = C$$

So B is unique

↳ inverse of A may or may not exist

If A is non singular $\Rightarrow$ inverse of A exists and is unique $\{$ inverse of $A = A^{-1}\}$

Remark :-

- $\frac{A}{B}$ Does not Exist / so If $AB = AC$ $\not\Rightarrow B = C$

Ex $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- $A = (a_{ij})_{n\times n}$

$\rightarrow$ You can't write A^{-1} Before you make sure That A actually has ~~a~~ an inverse

- If A, B are $n\times n$ - non singular Matrices

① - Is $A+B$ nonsingular (In general No)

take : $A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\therefore II = I = II$

I is non - singular , $I^{-1} = I$

$B = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is non singular

Because $(-I)(-I) = I$

~~A~~

$-I$ is non - singular , $(-I)^{-1} = I$

$A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ No solution

② AB is non singular and $(AB)^{-1} = B^{-1}A^{-1}$ — A, B are non-singular

Proof:- Assume A, B are non singular

$$(AB)(B^{-1}A^{-1}) \overset{?}{=} I$$

$$A\,(BB^{-1})\,A^{-1} = (A\,I)A^{-1}$$
$$= AA^{-1} = I \quad ✓$$

Similarly :- $\underline{(B^{-1}A^{-1})} = \underline{(AB)} = I$

So AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

* If A is non singular and $AB = AC$ Then $B = C$ } important

Proof $AB = AC$
A^{-1} exists : Multiply by A^{-1} from left

$$(A^{-1}A)B = (A^{-1}A)C$$

$$IB = IC$$
$$B = C$$

③ If A, B are singular, is $A-B$ singular

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{non-Singular}} - \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Non-singular}} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{singular}}$$ NO

Give $A_{n \times n}$: Is A non-singular?
If yes, find A^{-1}

Try: $A(?) = I = (?)A$

1.4 is finished

A is nonsingular ($n \times n$)

Is A^T nonsingular?

$(A^T)(A^T)^{-1} =$

You can't write it

$$(A^T)(A^{-1})^T = (A^{-1}A)^T$$
$$= (I)^T = I$$

Similarly $(A^{-1})^T A^T = I$

A^T is non-singular
and $(A^T)^{-1} = (A^{-1})^T$

1.5: Elementary Matrices

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \end{pmatrix}$$ is nonsingular

$$I \begin{cases} \text{row operation I} \longrightarrow E_1 : \text{Elementary of Type I} \\ \text{row operation II} \longrightarrow E_2 : \text{Elementary of Type II} \\ \text{row operation III} \longrightarrow E_3 : \text{Elementary of Type III} \end{cases}$$

- E is elementary of Type I if it can be obtained from I_n by applying row operation I <u>once</u>

same as R.O. II and R.O. III

Ex: $E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim$ Elementary of Type III

$$\uparrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow 2R_1 + R_2$$

$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is elem of Type I $\xleftarrow{R_1 \Leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ Not elementary