lec 9 :-

Theorem (Main Result) :-

let A be $n \times n$ - matrix then the following statements are equivalent :-

1) A is unsingular

2) The system $Ax = 0$ has only the zero solution

يعني نعمل operations على A وتوصل لـ I_n

3) A is row equivalent

Proof :-

1- ① → ② . يعني الاستنتاج رقم ① يوصلني إلى الاستنتاج رقم ②

Assume A is non-singular (A^{-1}) exists

let $\hat{x}$ be the solution to $Ax = 0$

$$\rightsquigarrow \underset{n\times n}{A}\,\underset{n\times 1}{\hat{x}} = \underset{n\times 1}{0}$$

Multiply by A^{-1} from left

$$A^{-1}(A\hat{x}) = A^{-1}(0)$$

$$I\hat{x} = 0$$

$$\boxed{\hat{x} = 0}$$

So $Ax = 0$ has only the zero solution

2) ② $\Longrightarrow$ ③

Assume $Ax=0$ has only the zero solution
let U be the reduced row echelon form of A (Show $U=I$)

$Ax=0$ & $Ux=0$ are equivalent
$\Longrightarrow$ $Ux=0$ has only the zero solution

$(A|0) \longrightarrow (U|0)$

$(U|0)$

$$U = \begin{pmatrix} & & \\ & & \\ 0 & \cdots & 0 \end{pmatrix}$$

If U has a row of zeros
\# of leading variables $< n$ (number of variables)

$\Longrightarrow$ There exist free variables
$\rightsquigarrow$ $Ux=0$ has infinite number of solutions

$\rightsquigarrow$ U has no row of zeros

$$U = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \Longrightarrow U = I$$

So $A \approx I$

3) ③ $\longrightarrow$ ①

Assume $A \approx I$

$\rightarrow$ There are elementary matrices $E_1 \text{---} E_k$

s.t $A = E_k \text{---} E_1 I = E_k \text{---} E_2 E_1$

each EI is non-singular (result)
also product of non-singular matrices is non singular

So :- A is nonsingular

and $A^{-1} = (E_k \text{---} E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \text{---} E_k^{-1}$

Method to compute A^{-1} :-

If A is non-singular $\Longrightarrow A \approx I$

$I \approx A$ So There exist elementary matrices $E_1 \text{---} E_k$ s.t $I = E_k \text{---} E_1 A$

$(I = E_k \text{---} E_1 A) \times A^{-1}$

$A^{-1} = (E_k \text{---} E_1 A) A^{-1}$

$A^{-1} = E_k \text{---} E_1 I$

Ex :- $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{pmatrix}$ is A nonsingular if yes find A^{-1}

after reduction

$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{pmatrix}$. I or Not I

$I \downarrow$ non singular $\downarrow$ نعمل ال row operation على I لنتوصل إلى A^{-1}

Not $I \downarrow$ singular.

لذلك اقوم بعمل ال reduction للمصفوفتين معاً

$$A = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$A = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -1 & 1 & 0 \\ 0 & -5 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -5 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} & -\frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & -1 & \frac{3}{5} \end{array}\right)$$

① $\frac{2}{3}R_3+R_2$ ↓ ② $-\frac{1}{3}R_3+R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{15} & 1 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{15} & -1 & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & -1 & \frac{3}{5} \end{array}\right)$$

$$I \Rightarrow A^{-1} = \begin{pmatrix} \frac{4}{15} & 1 & -\frac{1}{5} \\ \frac{3}{15} & -1 & \frac{2}{5} \\ -\frac{1}{5} & -1 & \frac{3}{5} \end{pmatrix}$$

A is non-singuler and

$\rightsquigarrow AA^{-1} = I$ (check)

To Sum Up :-

A is nonsinguler ? A^{-1} ?

$$(A|I) \xrightarrow{E.R.O} \begin{cases} \xrightarrow{A\ Non.S} (I|A^{-1}) \\ \xrightarrow{A\ Sing} \left(\begin{array}{c|c} \cancel{A \neq I} & \\ 0 \;\cdots\; 0 & X \end{array}\right) \end{cases}$$

$$\underset{n \times n}{A} X = 0$$

$A_{n\times n}\, X = 0$, $\underset{n\times n}{A}\, x = b$

① If $\underset{n\times 1}{\hat{x}}$, $\underset{n\times 1}{\hat{y}}$ are solutions to $Ax=0$ $\left(\begin{matrix} A\hat{x}=0 \\ A\hat{y}=0 \end{matrix}\right)$

Is $\hat{x}+\hat{y}$ a solution ?

$A(\hat{x}\pm\hat{y}) = A\hat{x} \pm A\hat{y}$
$= 0 \pm 0 = 0$

So <u>Yes</u> $\hat{x}\pm\hat{y}$ is a solution to $Ax=0$

② If $\hat{z}$, $\hat{w}$ are solutions to $Ax=b$
& $b\neq 0$

Is $\hat{z}+\hat{w}$ a solution ?

$A(\hat{z}+\hat{w}) = A\hat{z} + A\hat{y}$
$= b + b = 2b$

So <u>No</u> $\hat{z}+\hat{w}$ is not a solution

And $\hat{z}-\hat{w}$ is a solution to <u>$Ax=0$</u>

Theory :- The system $\underset{n\times n}{Ax}=b$ has unique solutions if & only if A is nonsingul

Assume $Ax=b$ has a unique solution
(A is non Singular)

- Assume A is singular, (Conculate a Contradiction)
by Theorem $\Rightarrow Ax=0$ has non zero solutions

$\Rightarrow$ There exist $\hat{x} \neq 0$ s.t $A\hat{x}=0$

$\Rightarrow$ So $\hat{y}+\hat{x}$ is a sol to $Ax=b$

$\rightsquigarrow Ax=b$ has at least two solutions
$(\hat{y}, \hat{y}+\hat{x})$ where $\hat{y} \neq \hat{y}+\hat{x}$
since $\hat{x} \neq 0$

A Contradiction $\rightsquigarrow$ A is nonsingular

- Assume A is non-singular (A^{-1} exists)
Solve $(Ax=b)$

$\downarrow$

$$A^{-1}(Ax) = A^{-1}(b)$$
$$x = A^{-1}b \quad \text{one solution}$$

Ex: Solve :-
$$x_1 + 2x_2 - x_3 = 1$$
$$x_1 - x_2 + x_3 = 4$$
$$2x_1 - 2x_2 + 3x_3 = 5$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{4}{15} & 1 & \frac{-1}{15} \\ \frac{3}{15} & -1 & \frac{2}{5} \\ \frac{-1}{5} & -1 & \frac{2}{5} \end{pmatrix}$$

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A is non singuler

Unique Solution $X = A^{-1}b$

$$= \begin{pmatrix} \frac{4}{15} & 1 & \frac{-1}{15} \\ \frac{3}{5} & -1 & \frac{2}{5} \\ \frac{-1}{15} & -1 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

$-2 \quad \frac{-1}{15}$

$$= \begin{pmatrix} \frac{16}{15} \\ \frac{7}{15} \\ -31 \end{pmatrix}$$