# Proofs of chapter 1 :-

* $A_{n\times n}$ is nonsingular And B is it's inverse

B is unique

Assume B, C are inverses of A

$AB = I = BA \qquad AC = I = \boxed{CA}$

Show That $B = C$

$\underline{B} = IB = (CA)B = (CA)B = C(AB) = \underline{CI}$

$B = CI$

$\hookrightarrow \boxed{B = C}$ ✓

* $A_{n\times n}$, $B_{n\times n}$ are nonsingular matrices

Show That AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$(AB)(B^{-1}A^{-1}) \overset{?}{=} I$

$A(BB^{-1})A^{-1} = AIA^{-1}$
$= AA^{-1} = I$

So AB is nonsingular & $(AB)^{-1} = B^{-1}A^{-1}$

* If A is non singular & $AB = AC$ Then $B = C$

$(A^{-1})AB = (A^{-1})AC$

$\boxed{B = C}$ ✓

* A is nonsingular, $A^T$ is nonsingular and $(A^T)^{-1} = (A^{-1})^T$

$(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$

* $E$ is elementary Then $E$ is non singular & $E^{-1}$ is elementary of the same Type

• If $E$ is Type I

$$\underline{I} \rightsquigarrow R_i \leftrightarrow R_j \rightsquigarrow \underline{E} \rightsquigarrow R_i \leftrightarrow R_j \rightsquigarrow \underline{I}$$

Remember:- $EA = B$
↓
Same as
↓
$EE = I$

• هنالكملة معناها اذا ضرب $A$ بـ $E$
من اليسار فالجواب هو مصفوفة $B$
وكأنني طبقت على $A$ العمليات الصفوية
التي طبقتها على $I$ لأصل الى $E$

• If $E$ is Type II

$I \rightsquigarrow \alpha R_i \rightsquigarrow E$ $\quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix}$

$I \rightsquigarrow \frac{1}{\alpha} R_i \rightsquigarrow F$ $\quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{\alpha} \end{pmatrix}$

$EF = I$

So $E$ is non singular & $F = E^{-1}$

• If $E$ is Type III

$I \rightsquigarrow cR_i + R_j \rightsquigarrow E$ $\quad \begin{pmatrix} 1 & & \\ c & & 1 \end{pmatrix}$

$I \longrightarrow -cR_i + R_j \rightsquigarrow F$ $\quad \begin{pmatrix} 1 & & \\ -c & & 1 \end{pmatrix}$

$EF = FE = I$

So $E$ is nonsingular

* if $A \approxeq B$ Then $B \approxeq A$

row equivalent

$$A = E_k \text{——} E_2 E_1 B$$

all elementary

$$E_k^{-1} A = E_{k-1} \text{——} E_2 E_1 B$$

$$E_{k-1}^{-1} E_k^{-1} A = E_{k-2} \text{----} E_2 E_1 B$$

$$\downarrow$$

$$\underbrace{E_1^{-1} E_2^{-1} \text{---} E_k^{-1}}_{\text{elementary}} A = B \qquad \text{so } A \approxeq B$$

* if $A \approxeq B$ & $B \approxeq C$ Then $A \approxeq C$

$$A = E_k \text{——} E_1 B$$

$$C = E_k \text{——} E_1 B$$

$$B = F_l \text{——} F_1 C$$

$$A = \underbrace{E_k \text{--} E_1 ( F_l \text{--} F_1}_{\text{elementary}} C)$$

$$A \approxeq C$$

* Proof of main Result

Proof of (2) :- The system has ($Ax = 0$) only zero solution if A is nonsingular

- A is non singular

$$\underset{n \times n}{A} \underset{n \times 1}{\hat{x}} = 0_{n \times 1} \rightsquigarrow (A^{-1}) A \hat{x} = (A^{-1}) 0$$

$$\hat{x} = 0$$

Proof of ③:-

$U$ is the reduced Row echelon form of $A$

- $Ax=0$ and $Ux=0$ are equivalent
- $Ux$ has only the zero solution

No assuming There is a zero row in $U$

So # leading variables $< n$

So There is free variables

So infinite solutions

This means That $U$ has no row ... zeros

So $U=I$ $\leadsto$ $E \cdots A = I$

and $A \approxeq I$

Proof of ① By ③:-

Assume $A \approxeq I$

$$A = E_k \cdots E_1 I = E_k \cdots \underbrace{E_2 E_1}_{\text{nonsingular}}$$

And Each product of is nonsingular

So $A$ is non singular

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

* $A_{n\times n}X = 0$ $\quad\quad$ $A_{n\times n}X = b$

$\hat{x}, \hat{y}$ are solutions to $Ax = 0$

$\hat{x} \pm \hat{y}$ is a solution :-

$$A(\hat{x} \pm \hat{y}) = A\hat{x} \pm A\hat{y} = 0 \pm 0 = 0$$

$\hat{z}, \hat{w}$ are solutions to $Ax = b$

$A($ [illegible] $) = A\hat{z} + A\hat{y} = 2b$

So it's not a solution

* $A_{m\times n}$

$A^T A$ and $A A^T$ are possible

$$A = \begin{pmatrix} \quad \end{pmatrix}_{m\times n} \longrightarrow A^T = \begin{pmatrix} \quad \end{pmatrix}_{n\times m}$$

$AA^T$ :- is valid

$$A^T = \begin{pmatrix} \quad \end{pmatrix}_{n\times m} \longrightarrow A = \begin{pmatrix} \quad \end{pmatrix}_{m\times n}$$

$A^T A$ is valid

* A,B are n×n symmetric matrices $A\begin{pmatrix} a_1 & a \\ a & a_2 \end{pmatrix}$ $B = \begin{pmatrix} b_1 & b \\ b & b_2 \end{pmatrix}$

$$C = A+B = \begin{pmatrix} a_1+b_1 & a+b \\ a+b & a_2+b_2 \end{pmatrix} \rightarrow \text{symmetric}$$

$D = A^2$ $\boxed{A^T = A}$

$D^T \stackrel{?}{=} D$

$D^T = (A^2)^T$

$= (A^T A^T)^2$

$= A^2 = D$ so $\rightarrow$ symmetric

$E = AB$

$E \stackrel{?}{=} E^T$

$E^T = (AB)^T$

$= B^T A^T$

$= BA \neq AB \longrightarrow$ Not symmetric

* $F = ABA$

$F \stackrel{?}{=} F^T$

$F^T = (ABA)^T$

$= ((AB) \cdot A)^T$

$= A^T (AB)^T$

$= A^T B^T A^T$

$= ABA = F \longrightarrow$ symmetric

$\therefore G = AB+BA$

* $H = AB - BA$

$H^T = B^T A^T - A^T B^T$

$= BA - AB \neq AB - BA \rightarrow$ Not symmetric

* $AB = A$ and $B \neq I$ Then $A$ must Be singuler

Assuming $A$ is nonsinguler ($B$ $A_{n \times n}$ )

$A^{-1} AB = A^{-1} A$

$B = I \rightsquigarrow$ Contradiction So $A$ is singular

* Show that $(A^{-1})^{-1} = A$

$A A^{-1} = I$

↳ This means that $A$ is The inverse of $A^{-1}$

And $A^{-1}$ is the inverse of $A$.

So :- $(A^{-1})^{-1} = A$

* $A$ is nonsingular Then $A^T$ is nonsinguler.

$(A^T)^{-1} = (A^{-1})^T$

$A A^{-1} = I$

$(A A^{-1})^T = (I)^T$

$(A^{-1})^T A^T = I$ $(A^T)^{-1}$ نضرب الطرفين ب

$(A^T)^{-1} (A^{-1})^T (A)^T = I (A^T)^{-1}$

$I (A^{-1})^T = (A^T)^{-1} I$

$(A^{-1})^T = (A^T)^{-1}$

* $A_{n\times n}$, $x, y$ vectors

If $\underline{Ax = Ay}$, $x \neq y$ Then $A$ must be singular

Assume $A$ is non-singular

$Ax = Ay$

$(A^{-1})Ax = (A^{-1})Ay$

$x = \underline{y}$ (contradiction) so $A$ is singular

* $A_{n\times n}$, $A^2 = 0$ $\Longrightarrow$ $I - A$ nonsingular

And $(I-A)^{-1} = (I+A)$

$(I-A)(I+A) = I^2 - \cancel{AI}\ \cancel{IA} - A^2$

$= I$

so $(I+A)$ is The inverse of $I - A$

* show That $I - A$ is an idempotent matrix

$(I-A)^2 = (I-A)(I-A)$

$= I^2 - IA - AI + A^2$

$= I - 2A + A^2 \rightarrow = A$

$= I - 2A + A$

$= I - A$

* $D_{m\times n}$ diagonal matrix, $X$ is nonsingular
Show That $A = XDX^{-1}$ is also idempotent

$$A = (XX^{-1})D = D$$

$$A^2 = (XDX^{-1})(XDX^{-1})$$
$$= D^2 = A$$

* $A_{m\times n}$, show That $A^TA$ and $AA^T$ are symmetric

let's assume that $B = A^T_{n\times m} A_{m\times n}$ and $C = A^{T}_{m\times m} A^T_{m\times n}$

$$(B_{m\times n})^T = (A^TA)^T = A^T(A^T)^T = A^TA = B$$
So its symmetric

$$(C_{m\times m})^T = (AA^T)^T = (A^T)^T A^T = AA^T = C_{m\times m}$$
So it's symmetric

* $A, B_{n\times n}$ Symmetric, $AB = BA$ only if $AB$ is also symmetric

$$(AB)^T = B^TA^T = BA$$
since $AB$ is symmetric Then $(AB)^T = AB$
So $AB = BA$

* $B = A + A^T$ $B$ is symmetric $C = A - A^T$ $C$ is Skew symmetric

$$(B)^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$$
$$(C)^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$
$$= -C$$

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* Product of two Elementary matrices ~~a~~ is not an elementary matrice

* The transpose of an Elementary matrix is The same as it

Type I :- E is symmetric
$$\text{so } E^T = E$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
symmetric

Type II :- E is symmetric
$$\text{so } E^T = E$$

Type III :- only C is different and appears outside The diagonal. So in $E^T$ only C will appear outside The diagonal Too so It's Type III

* $A, B_{n \times n}$ $\quad C = A - B$

$Ax_0 = Bx_0 \quad x_0 \neq 0 \quad$ C must be singuler.

assuming C is non singuler.

$$Ax_0 - Bx_0 = 0$$
$$\cancel{A(x_0}$$
$$x_0(A - B) = 0$$
$$Cx_0 = 0 \quad \text{(Main Result 2)}$$
$x_0 \neq 0$ So Contradiction :-
C is singuler

* $A, B_{n \times n}$ $\quad C = AB \quad$ if B is singuler Then C is singuler

We Know That a matrix M is nonsinguler only if $Mx = 0$ and $Bx = 0$ has a solution $x \neq 0$ only if it's singuler

unique solution $x = 0$

so $Cx = ABx \rightarrow = 0$

$Cx = 0$ where $x \neq 0$ so $C$ must be singular

* If $A$ is symmetric Then $A^{-1}$ is symmetric

$$AA^{-1} = I$$

$$(AA^{-1})^T = (A^{-1})^T (A^T) = I$$

$$(A^{-1})^T (A^T) = I$$ نضرب $(A^T)^{-1}$ من الطرفين

$$(A^{-1})^T (A^T)(A^T)^{-1} = I (A^T)^{-1}$$

$$(A^{-1})^T = (A^T)^{-1}$$

now

$$A = A^T$$

$$A^{-1} = (A^T)^{-1} = (A^{-1})^T$$

$$(A^{-1}) = (A^{-1})^T$$ so it's symmetric

* If $A \cong B$ Then $B \cong A$

$$A = (E_{k-1} \text{---} E_1) B$$

$$(E_{k-1})^{-1} A = (E_{k-2} \text{---} E_1) B$$

$$(E_{k-1})^{-1} (E_{k-2})^{-1} A = (E_{k-3} \text{---} E_1) B$$

$$(E_{k-1})^{-1} (E_{k-2})^{-1} \text{---} E_1^{-1} A = B$$

So $B$ is row equivalent to $A$

* $A \cong B$ and $B \cong C$

Then $A \cong C$

$A = E_{k-1} \cdots E_1 B$

$B = F_{R-1} \cdots F_1 C$

$$A = \underbrace{(E_{k-1} \cdots E_1)(F_{R-1} \cdots F_1)}_{\text{elementory}} C$$

so $A \cong C$

* $B \cong A$ only if $M$ is nonsinguler where $B = MA$

$B = E_K \cdots E_1 A$

assumig $M = E_K \cdots E_1$ it's a product of nonsingular matrices so it's nonsingular.

Since $M$ is nonsinguler $M \cong I$

$M = E_K \cdots E_1 I = E_K \cdots$

so $B = E_K \cdots E_1 A$ so $B \cong A$

* if $A \cong I$ And $AB = AC$ Then $C$ must equal $B$

(Back to main result)

if $A \cong I$ Then it's nonsinguler. So There is $A^{-1}$

Where $AA^{-1} = I$

$AB = AC$

$A^{-1}AB = A^{-1}AC$

$IB = IC \Rightarrow B = C$