$\ast$ $A_{n\times n}$, $\det(\alpha A) = \alpha^n \det(A)$

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & & \\ & & \alpha - & \\ -- & - & & \alpha \end{pmatrix}$$

$$|\alpha I| = \alpha^n$$

$$\det(\alpha I A) = \det(\alpha I)\det(A)$$
$$= \alpha^n \det(A)$$

Proofs of chapter 2

$\ast$ A nonsingular

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$
$$\det A \det A^{-1} = \det(I)$$
$$\det A \det A^{-1} = 1$$
$$\det A^{-1} = \frac{1}{\det A}$$

$\ast$ A,B $n\times n$, AB is nonsingular only if A and B are nonsingular

Assume A,B are nonsingular

$$\cancel{(AB)^{-1} = B^{-1}A^{-1}}$$

$$AB\,(B^{-1}A^{-1}) \stackrel{?}{=} I$$

$$A\underbrace{BB^{-1}}_{I}A^{-1} = AIA^{-1} = AA^{-1} = I$$

So AB is singular only if A,B are nonsingular.

* A, B $n \times n$ matrices

$AB = I$, Then $BA = I$

* $A_{n \times n}$, skew symmetric if n is odd Then A must be singular

$A = -A$

$\det(A) = \det(-1 \times A)$

$= (-1)^n \det(A)$ if n is odd Then

$\rightarrow \det(A) = -\det(A)$ This is True only if $\det(A) = 0$

So A is singular.

* $\det(\operatorname{adj} A) = \det(A)^{n-1}$ $A_{n \times n}$ nonsingular.

Remember

$A \operatorname{adj}(A) = \det(A) I$

$\det(A \operatorname{adj}(A)) = \det(\det(A) I)$

$\det(A)\det(\operatorname{adj}(A)) = \det(A)^n \underline{\det(I)} \rightarrow \det(I) = 1$

$\det(\operatorname{adj}(A)) = \dfrac{\det(A)^n}{\det(A)}$

$\det(\operatorname{adj}(A)) = \det(A)^{n-1}$

* A nonsingular $\rightarrow$ adj (A) nonsingular and

$$\operatorname{adj}(A)^{-1} = \det(A^{-1})\, A \quad = \operatorname{adj}(A^{-1})$$

① $\rightarrow$ $A \operatorname{adj}(A) = \det(A)\, I$

Put A^{-1} instead of A

$$A^{-1} \operatorname{adj}(A^{-1}) = \det(A^{-1})\, I$$

$$AA^{-1} \operatorname{adj}(A^{-1}) = A \det(A^{-1})\, I$$

$$\operatorname{adj}(A^{-1}) = \det(A^{-1})\, A$$

---

② $A \operatorname{adj}(A) = \det(A)\, I$

$$\frac{1}{\det(A)} A \operatorname{adj}(A) = \frac{1}{\cancel{\det(A)}} \cancel{\det(A)}\, I$$

$$\frac{A}{\det(A)} \operatorname{adj}(A) = I$$

$$A \underbrace{\operatorname{adj}(A)^{-1} \operatorname{adj}(A)}_{I} = \det(A) \operatorname{adj}(A)^{-1}$$

$$\cancel{\frac{1}{\det(A^{-1})}} \leftarrow \frac{A}{\det(A)} = \operatorname{adj}(A)^{-1}$$

$$A \det(A^{-1}) = \operatorname{adj}(A)^{-1}$$

so ① = ②

* If $\det(A) = 1$ Then

$$\operatorname{adj}(\operatorname{adj}(A)) = A$$

Replace ~~as~~ A by adj(A) in ①

$$\operatorname{adj}(A) \operatorname{adj}(\operatorname{adj}(A)) = \underbrace{\det(\operatorname{adj}(A))}\, I \qquad \text{is } \cancel{\det A}\ A$$

$$\operatorname{adj}(A)^{-1} \operatorname{adj}(A) \operatorname{adj}(\operatorname{adj}(A)) = 1 = \underbrace{\det(A)^{n-1}}\, I \quad \operatorname{adj} A^{-1}$$

$$\operatorname{adj}(\operatorname{adj} A) = A$$