Chapter 3: Vector spaces

A matrix commute with another matrix $\Rightarrow AB = BA$

3.1:- Definitions and Examples:-

- A vector space:- V : a set with two well defined operations ① → Addition
  ② → Scalar Multiplica
  for all $V_1, V_2 \in V$
  → Where: $V_1 + V_2 \in V$: for all $v \in V$
  and α is a scalar where $\alpha V \in V$

- Conditions of vector space:-
  In Addition to ① and ②, A set is called a vector space if:-

1- $V_1 + V_2 = V_2 + V_1$ for all $V_1, V_2 \in V$
2- $V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$ for all $V_1, V_2, V_3 \in V$
3- There exists a special element $\vec{0}$ (zero vector) That satisfies $\vec{0} + V = \vec{V}$ for all $v \in V$
4- for every $v \in V$, There exists an element $-v \in V$ that satisfies :- $(\vec{V}_i + -\vec{V}) = \vec{0}$
5- $\alpha(V_1 + V_2) = \alpha V_1 + \alpha V_2$, for all $V_1, V_2 \in V$ and α is a scalar
6- $(\alpha + \beta)V = \alpha V + \beta V$ for all $v \in V$, α, β scalars
7- $(\alpha\beta)V = \alpha(\beta V)$ for all $v \in V$, α, β scalars
8- $1 . V = V$ for all $v \in V$

- Basic Examples in this Chapter :-

① - $\mathbb{R}^n \Rightarrow \begin{pmatrix} x \\ \vdots \\ x_n \end{pmatrix}$ Ex:- $\mathbb{R}^2 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

② $\mathbb{R}^{m\times n} \rightsquigarrow$ Matrix of size $M\times n$

③ $C[a,b] = \{ f(x) : f(x)$ is a Continuous function on $[a,b] \}$

④ P_n :- set of all polynomials of degree ~~to~~ less Than n

- Proof of each one is in the "Proofs of chapter Three files"

- Additional proprieties of a vector space

let V be a Vector Space :-

→ 1) $\boxed{0V = \vec{0}}$, for all $v \in V$

→ 2) If $v, w \in V$ and $\boxed{v + w = \vec{0}}$, Then $\boxed{w = -v}$

→ 3) $\boxed{(-1)v = -v}$, for all $v \in V$

3.2 : Subspaces

- S is a subspace of V if:-
  - $S \subseteq V$
  - $S \neq \phi$
  - for all $s_1, s_2 \in S$, $s_1 + s_2 \in S$
  - for all $s \in S$, α scaler : $\alpha s \in S$

- Remark :- if $0 \notin S$ Then S is not a subspace of V (condition ② is not Satisfied)

- Null space :- $N(A)$ ⇝ It's a subspace of A
  matrix
  - $N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$:- The set of all solutions to $Ax = 0$

- Span :-

* let V be a vector space $V_1, V_2, ---, V_n \in V$
The set of all linear combinations of $V_1, V_2, -- V_n$ is called a span of $V_1, V_2, ---, V_n$

$$\text{Span}(V_1, ---, V_n) = \{ \alpha_1 V_1 + \alpha_2 V_2 + ---- + \alpha_n V_n \}$$

$\alpha_i \in \mathbb{R}$ (scalar)

* A span is a subspace of V so:-
1) Span $\neq \phi$ (There is $0 = 0V_1 + 0V_2 + --- + 0V_n$
2) $W, U \in \text{span} \Rightarrow W + U \in \text{Span}$
3) and if $W \in \text{span} \Rightarrow \alpha (W) \in \text{span}$

- How to know if a set is a spanning set for V or not?

  * You solve the system

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} V_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} V_2 \end{pmatrix} + \text{——} + \alpha_n \begin{pmatrix} V_n \end{pmatrix}$$

  * If it's consistent for all values then $\{V_1, \text{—}, V_n\}$ is a spanning set

3.3: linear Independance

let $V_1, \ldots, V_n \in V$

The system $C_1V_1 + C_2V_2 + \cdots + C_nV_n = 0$

has only the zero solution → linearly independant

has non zero solution → linearly dependant

• To Know if a set is linearly dependant or independant :-

solve $c_1(V_1) + c_2(V_2) + c_3(V_3) = 0$

↳ if it has only the zero sol Then it's L.I

↳ if it has other solutions Then it's L.D

* We can Write V_1 a l. Combination of the other $V_2, \ldots, V_n$ if They were L.D

* Cases for sets :-

1- $\{V_1, \ldots, V_n\}$ is a spanning set for V : $v \in V$

Then $\{V_1, \ldots, V_n, V\}$ is still a spanning set

and $\{V_1, \ldots, V_{n-1}\}$ may or may not be a sp. set

2- $\{V_1, \ldots, V_n\}$ is L.I

Then $\{V_1, \ldots, V_n, V\}$ may or may not be L.I

and $\{V_1, \ldots, V_{n-1}\}$ are still L.I

3- If $\{V_1, V_2, \dots, V_n\}$ are L.D

Then $\{V_1, \dots, V_n, V\}$ are L.D

and $\{V_1, \dots, V_{n-1}\}$ may or may not be L.D

Wronskian

If $f_1, f_2, \dots, f_n$ are functions in $C^{n-1}[a,b]$ Then

Wronskian :- $$\omega[f_1, \dots, f_n](x_0) = \begin{bmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & & & \\ f_1^{n-1}(x) & f_2^{(n-1)}(x) & \dots & f_n^{n-1}(x) \end{bmatrix}$$

→ if $W \neq 0$ → functions are L.I

→ if $W = 0$ → Test fails

- Note: it works for Polynomials Because They are functions

Theory :-

* If $V_1, V_2, \dots, V_n$ $\quad V_n \in \mathbb{R}^n$ Then $V_1, \dots, V_n$ are L.I iff the Matrix $X = (V_1 \ V_2 \ \dots \ V_n)_{n \times n}$ is non singuler

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Theory :-

* If $w \in$ span $(V_1, \dots V_n)$, we can write w uniquely as a linear combination of $V_1, \dots, V_n$ iff $V_1, \dots, V_n$ are linearly Ind

W (Unique L.C) $\Longleftrightarrow$ $V_1, \dots, V_n$ L.I

3.4: Basis and Dimensions

- let $V_1, \dots, V_n \in V$

$\{V_1, \dots, V_n\}$ is a basis ~~set~~ for V if

1) $\{V_1, \dots, V_n\}$ is a spanning set for V

2) $V_1, V_2, \dots, V_n$ are L.I

- let V be a vector space. if V has a basis with n vectors $\{V_1, \dots, V_n\}$
- Dimension of V is n

$\dim(V) = n$

- If V has no basis

$\dim(V) = \infty$

* standard basis for $\mathbb{R}^n$

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

* standard basis for $\mathbb{R}^{m \times n}$

$$\left\{ \begin{bmatrix} 1 & 0 & \cdots \\ \vdots & & \\ & \cdots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \cdots & \\ 0 & & \\ \vdots & \cdots & 1 \end{bmatrix} \right\}$$

* $\dim(\mathbb{R}^n) = n$

* $\dim(\mathbb{R}^{m \times n}) = mn$

Note:-

- $C[a,b]$ has no basis so $\dim(C) = \infty$
- $\dim\{0\} = 0$
- $V = \{0\}$ has no basis

Result

A set of $V_1, \dots, V_n$

$\dim(V) = n$, $n \geq 1$

$\dim(V) > n$: set is not a basis

$\dim(V) < n$: set is not a basis

→ If $\{V_1, \dots, V_n\}$ is a sp. set Then $V_1, \dots, V_n$ are L.I and so it's a basis

→ If $\{V_1, \dots, V_n\}$ are L.I Then They form a sp. set and so it's a basis

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- let V be a vector space :-

$\dim(V) = n > 0$

- set with less than n vectors can't be a spanning set
- set of more than n vectors in L.I

- Any L.I set with less than n vectors can be extended to a basis
- Any sp. set with more than n vectors can be paired down to a basis

Extending

- $\{v_1, \ldots, v_n\}$ are L.I
- $\dim(V) = n \quad s < n$
- Choose $w_1 \notin \operatorname{span}(v_1, \ldots, v_s)$

and $\{v_1, \ldots, v_n, w\}$ are L.I

$\boxed{s+1}$

$\boxed{s+1 = n}$

- $\{v_1, \ldots, v_n, w\}$ is a basis

$\boxed{s+1 < n}$

- Choose w_2 s.t $w_2 \notin \operatorname{span}(v_1, \ldots, v_n, w_1)$

$\{v_1, \ldots, v_s, w_1, w_2\}$ L.I

$= \boxed{s+2}$

$\boxed{s+2 = n}$ The set is a basis

$\boxed{s+2 < n}$ The set is not a basis so Continue

Reducing

- You have a spanning set to reduce it into a basis

* Remove one that can be written as a l.c of the others

How? Solve $v_1 c_1 + \ldots + v_n c_n = 0$

The one That it's coefficient not equal to zero

Remove one at a time

If $\dim(V) = n$ a S is a subset of V

Then

If S spans V $\longrightarrow$ L.I

If S is L.I $\longrightarrow$ Sp