Proofs of Chapter 3

Show That for V (a vector space)

$\vec{0 \cdot V} = \vec{0}$

let $v \in V$

$V = 1 \cdot V \longrightarrow$ (8 conditions)

$= (1+0) \cdot V$

$V = 1 \cdot V + 0 \cdot V$

$\boxed{V = V + 0 \cdot V}$

Now :- $\vec{0} = V + (-V)$

$\vec{0} = V + 0 \cdot V + (-V)$

$\vec{0} = V + (-V) + 0 \cdot V$

$\vec{0} = 0 \cdot V + 0$

$\boxed{\vec{0} = 0 \cdot V}$ ✓

Show That if $V, W \in V$ and $V + W = 0$

Then $V = -W$

let $V + W = 0$

$-V = -V + \vec{0}$

$= -V + (V + W)$

$-V = (-V + V) + W$

$-V = 0 + W = W$

so $-V = W$

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Show That $(-1)(V) = -V$

$$(-1)(V) + V = (-1)(V) + 1.V$$
$$= ((-1)+(1))V$$
$$= 0.V = 0$$

So $(-1)(V) + V = 0$

$$\boxed{(-1)(V) = -V}$$

If S is a subset of V, $0 \notin S$ Then S is not a Subspace for V

let $s \in S$

consider $\alpha.s = (0)(s) = 0 \notin S$

So Condition ② is not satisfied

So S is not a Subspace

$N(A)$ is a subspace of $\mathbb{R}^n$ (solutions to $Ax=0$)

① $N(A) \neq \phi$ $\quad x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in N(A)$

② $Av=0$, $Aw=0$

$A(v+w) = Av+Aw = 0+0 = 0 \in N(A)$

③ $\alpha v \in N(A)$

since $A(\alpha v) = \alpha A(v)$

Span $(V_1, V_2, \ldots, V_n)$ is a subspace of V

① span $\neq \emptyset$

$0 = 0V_1 + 0V_2 - - + 0V_n \quad \in S$

② $W, U \in$ span $(V_1, \ldots, V_n)$

$W = \alpha_1 V_1 + \alpha_2 V_2 \ldots$

$U = \beta_1 V_1 + \beta_2 V_2 \ldots$

$W + U = (\alpha_1 + \beta_1)V_1 + (\alpha_2 + \beta_2)V_2 + \ldots \in$ Span

③ $W \in$ span -- α scaler

$\alpha W = \alpha(\alpha_1 V_1 + \ldots + \alpha_n V_n)$

$= (\alpha\alpha_1)V_1 + \ldots + (\alpha\alpha_n)V_n$

$\in$ Span

If $V_1, V_2, \ldots, V_n \quad V_n \in \mathbb{R}^n$

Then $V_1, \ldots, V_n$ are L.I iff The matrix

$X = (V_1 \; V_2 \; \ldots \; V_n)$ is nonsingular

① $A = (V_1, V_2, \ldots, V_n)$ $\quad A(x=0)$

$x_1 a_1 + x_2 a_2 + \ldots x_n a_n$

$x_1 V_1 + x_2 V_2 + \ldots x_n V_n = 0$

it has only The zero Sol since

$V_1, \ldots, V_n$ are L.I

So $Ax = 0$ has only The zero Sol

So A is nonsingular

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② $A = (V_1\ V_2\ \text{——}\ V_n)$ is nonsingular

System $C_1V_1 + C_2V_2 + \text{——} + C_nV_n = 0$ is equivalent to $AC = 0$

$C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ but A is nonsingular.

$AC = 0$ has only The zero Solution

So $V_1, \text{——}, V_n$ are L.I

If $V_1, \text{---}, V_n \in V$

$W \in \text{Span}(V_1, \text{——}, V_n)$ Then we can write W uniquely as a linear combinations of $V_1, \text{—} V_n$ iff $V_1, \text{---}, V_n$ are linearly independent

① Assume $V_1, \text{—} V_n$ L.D / Assum W can be written uniquely

① Assume $V_1, \text{—} V_n$ are not all zeros

$\exists\ C_1, \text{---} C_n$ are not all zeros

s.t $C_1V_1 + \text{——} + C_nV_n = 0$

$W = \alpha_1V_1 + \alpha_2V_2 + \text{——} \alpha_nV_n$

$W + 0 = \alpha_1V_1 + \alpha_2V_2 + \text{——} + \alpha_nV_n + C_1V_1 + \text{——} + C_nV_n$

$= (\alpha_1 + C_1)V_1 + (\alpha_2 + C_2)V_2 + \text{———} (\alpha_n + C_n)V_n$

$$\left.\begin{array}{l} \alpha_1 + C_1 = \alpha_1 \rightarrow C_1 = 0 \\ \alpha_2 + C_2 = \alpha_2 \rightarrow C_2 = 0 \\ \vdots \\ \alpha_n + C_n = \alpha_n \rightarrow C_n = 0 \end{array}\right\} \text{uniqueness}$$

Contradiction So $V_1, \text{---}, V_n$ are L.I

② Assume $V_1, \text{---}, V_n$ are L.I

Assume $W = \alpha_1V_1 + \alpha_2V_2 + \text{——} + \alpha_nV_n$

$= \beta_1V_1 + \beta_2V_2 + \text{-----} + \beta_nV_n$

Show $(\alpha_1 = \beta_1, \text{——} \beta_n = \alpha_n)$

$\Rightarrow 0 = W - W \Rightarrow 0 = (\alpha_1 - \beta_1)V_1 + \text{——} + (\alpha_n - \beta_n)V_n$

But $V_1, \text{——} V_n$ are L.I

So $\beta_1 = \alpha_1$

$\beta_n = \alpha_n$