Chapter 6 :- 6.1 : Eigenvalues of a matrix

- A is a $n \times n$ matrix
- λ is a scaler called an eigenvalue
- X : non zero vector called an eigenvector corresponding to λ

$$Ax = \lambda x$$

Eigenvalue　　Eigenvector

* The system $(A-\lambda I)x = 0$

↳ has a non-zero solution (since $x \neq 0$)

⇒ So $(A-\lambda I)$ is nonsingular and $|A-\lambda I| = 0$

- Eigenvalues are the roots of ($\det(A-\lambda I) = 0$)
- Eigenvectors are the non-zero solutions to The system $(A-\lambda I)x = 0$

Eigenspace : ~ for each eigenvalue There is an eigenspace

~ The eigenspace of λ_0 is the Null space of $A-\lambda_0 I$

~ $E(\lambda_0)$ is a subspace of $\mathbb{R}^n$

~ Eigenvectors Does not form a subspace for $\mathbb{R}^n$ Because $X \neq 0 \Rightarrow 0 \notin \mathbb{R}^n$

But Eigenspace is a subspace

• Charachteristic Polynomial :- ~> of degree n (It has n roots)

$P(\lambda) = |A - \lambda I|$

• roots of $P(\lambda)$ are the eigenvalues for A (solve $P(\lambda) = 0$)

→ The roots can be Real numbers or Imaginary numbers

• roots can be different or multiplied

n = multiplicity + different

• distinct roots: Multiplicity not Counted

→ number of distinct eigenvalues $\leq n$

⚠ Remark :- If $\lambda = 0$ is an eigenvalue for A Then A is "singular"

Notes

[1] • If A is a real matrix (meaning All the elements are real numbers)

And $z = a + ib$ is an eigenvalue for A with eigen vector X

Then :- * $\bar{\lambda} = a - ib$ is also an eigenvalue

$\bar{X}$ is an eigenvector for $\bar{\lambda}$

Ex:- $X = \begin{pmatrix} 1 \\ i \end{pmatrix}$ Then $\bar{X} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\lambda = 1 + 2i$ $\qquad\qquad$ $\bar{\lambda} = 1 - 2i$

$\boxed{2}$ $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigenvalues

↪ $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \text{——} (\lambda - \lambda_n) = \det(A)$ if $\lambda = 0$

↪ Trace of (A) :- $Tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

$$= \sum_{i=1}^{n} a_{ii}$$

→ Sum of elements of A on the Main Diagonal

Similar Matrices :-

* Matrix $B_{n\times n}$ is similar to $A_{n\times n}$ if there exists a nonsingular matrix S

Such That :- $A = S^{-1} B S$ (And A is similar to B)

* If A is similar to B Then :-

$\boxed{1}$ They have The same Characteristic Polynomial

$\boxed{2}$ " " " " eigenvalues

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* How to Solve Problems ?

outline find the eigenvalues and the eigenspaces of A:-

1) h $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

$$|A - \lambda I| = \left| \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{pmatrix} \right| .$$

$$= (1-\lambda)\left[(3-\lambda)(-1-\lambda) - 5\right] = 0$$

$$= (1-\lambda)\left[-3-3\lambda+\lambda+\lambda^2-5\right] = 0$$

$$= (1-\lambda)\left[\lambda^2-2\lambda-8\right] = 0$$

$$= (1-\lambda)(\lambda-4)(\lambda+2) = 0$$

So:- $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -2$

① $\lambda_1 = 1$

$E(\lambda_1 = 1) = \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} , \alpha \text{ scalar} \right\}$

$A - I = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix}$ solve for zero

$$= \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{9}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$X_3 = 0$
$X_2 = 0$
$X_1 = \alpha$

② $\lambda_2 = 4$

$$E(\lambda_2 = 4) = \left\{ \begin{pmatrix} \beta \\ \beta \\ \beta \end{pmatrix}, \beta \text{ scalar} \right\}$$

$$A - 4I = \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{pmatrix}$$ Solve for zero

$$= \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = \beta$
$x_2 = \beta$
$x_1 = \frac{-3\beta}{-3}$
$= \beta$

③ $\lambda_3 = -2$ (same way)

Ex 14 (outline):-

$Tr(A) = 8$
$\det(A) = 12$
$A_{2\times2}$ Then eigenvalues = ??

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\underline{a_{11} + a_{22} = 8}$ *

$\underline{a_{11}a_{22} - a_{12}a_{21} = 12}$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

$|A - \lambda I| = 0$

$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$

$\underline{a_{11}a_{22}} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - \underline{a_{12}a_{21}} = 0$

$\underline{a_{11}a_{22} - a_{12}a_{21}} - \lambda(a_{11} \overset{*}{+} a_{22}) + \lambda^2 = 0$

$12 - 8\lambda + \lambda^2 = 0$

$\lambda_1 = 2, \quad \lambda_2 = 6$