6.3: Diagnolization

- A matrix $A_{n\times n}$ is called diagnolizable if A is similar to a diagonal matrix $D_{n\times n}$
  s.t :- $A = XDX^{-1}$

- If $\lambda_1, \lambda_2, ---, \lambda_K$ are distinct eigenvalues with eigenvect $x_1, x_2 -- x_K$ Then $x_1, x_2, ---, x_K$ are linearly independ

→ Maximum number of Linearly independant eigenvector = Sum of dimensions of all eigenspaces

Diagnolizable:- iff A has n linearly independant eigenvectors

eigenvectors (basis for each eigenspace)

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & --- & x_n \end{pmatrix}$$

where λ_1 is corresponded to x_1

$$D = \begin{pmatrix} \lambda_1 & 0 & --- & 0 \\ 0 & \lambda_2 & --- & 0 \\ \vdots & \vdots & \ddots & \lambda_n \end{pmatrix}$$

λ_2 is corresponded to x_2 and so on

Note:
Defective = Not diagnol

* X, D are not unique
* $A_{n\times n}$ has n distinct eigenvalues → Diagnolizable
  " " less Then n → May or May not be Diagnolizable

Remark:- To find $A^K = XD^KX^{-1}$

* How to solve problems?

1) If $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$

$$|A - \lambda I| = \left| \begin{pmatrix} 1-\lambda & 2 & -1 \\ 2 & 4-\lambda & -2 \\ 3 & 6 & -3-\lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 4-\lambda & -2 \\ 3 & 6 & -3-\lambda \end{vmatrix} = (1-\lambda)\left[(4-\lambda)(-3-\lambda) + 12\right] + 2\left[2(-3-\lambda) + 6\right] - \left[12 - 3(4-\lambda)\right]$$

$$= (1-\lambda)\left[\cancel{-12} - 4\lambda + 3\lambda + \lambda^2 + \cancel{12}\right] + \bar{2}\left[\cancel{-6} - 2\lambda \cancel{+6}\right] - \left[\cancel{12} - \cancel{12} + 3\lambda\right]$$

$$= (1-\lambda)\left[(\lambda^2 - \lambda)\right] + \bar{2}\left[-2\lambda\right] + -3\lambda$$

$$= (1-\lambda)(\lambda^2 - \lambda) + 4\lambda - 3\lambda$$

$$= \lambda^2 - \lambda - \lambda^3 + \lambda^2 + 4\lambda - 3\lambda$$

$$= 2\lambda^2 - \lambda^3$$

$$= \lambda^2(2 - \lambda)$$

$\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2$

$E(\lambda_2 = \lambda_1 = 0)$:-

Solve :- $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = \alpha$
$x_2 = \beta$
$x_1 = -2\beta + \alpha$

$\Rightarrow E(\lambda_1 = 0) = \left\{ \begin{pmatrix} -2\beta + \alpha \\ \beta \\ \alpha \end{pmatrix} \right\}$

Basis = $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$E(\lambda_3 = 2)$:-

Solve :- $\begin{pmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ 3 & 6 & -5 \end{pmatrix}$

$\begin{pmatrix} -1 & 2 & -1 \\ 0 & 6 & -4 \\ 0 & 12 & -8 \end{pmatrix}$

$\begin{pmatrix} -1 & 2 & -1 \\ 0 & 6 & -4 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \boxed{X_3 = \alpha}$

$6X_2 = 4\alpha$

$\boxed{X_2 = \frac{2}{3}\alpha}$

$-X_1 = -2X_2 + X_3$

$-X_1 = -2\left(\frac{2}{3}\alpha\right) + \alpha$

$= -\frac{4\alpha}{3} + \frac{3}{3}\alpha$

$\boxed{X_1 = \frac{\alpha}{3}}$

Basis $E(\lambda_3 = 2) = \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} \right\}$

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$X = \begin{pmatrix} -2 & 1 & \frac{1}{3} \\ 1 & 0 & \frac{2}{3} \\ 0 & 1 & 1 \end{pmatrix}$ Then find X^{-1}

4)a) find $B^2 = A$ where $A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$

$|A - \lambda I| = \left| \begin{pmatrix} 2-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} \right| = (2-\lambda)(-1-\lambda) + 2$

$= -2 - 2\lambda + \lambda + \lambda^2 + 2$

$= -\lambda + \lambda^2 = 0$

$= \lambda(-1+\lambda) = 0 \rightsquigarrow \lambda = 0,\ \lambda_2 = 1$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}$$

$$X^{-1} = -2 \begin{pmatrix} 1 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix}$$

$$(D)^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = X D^{\frac{1}{2}} X^{-1}$$

$$= \begin{pmatrix} -\frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}$$

If $\lambda_1 = 0$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$x_2 = \alpha \quad x_1 = -\frac{1}{2}\alpha$

$$E(\lambda_1 = 0) = \left\{ \begin{pmatrix} -\frac{1}{2}\alpha \\ \alpha \end{pmatrix} \right\}$$

If $\lambda_2 = 1$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$x_2 = \beta \ , \ x_1 = -\beta$