* skew-symmetric:
If $\boxed{A^T = -A.}$

symmetric: A
if $A^T = A$.

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix} \text{ is skew symmetric.}$$

$$A^T = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix} = -A$$

$$\begin{pmatrix} - & ? & ? \\ ? & - & \\ ? & & - \end{pmatrix}$$

16/1.3) $A_{n\times n}$ skew symmetric if $A^T = -A$.
If $A_{n\times n}$ is skew symmetric, then $\boxed{a_{ii} = 0, \ \forall i.}$
diagonal entries.

Assume A is skew-symmetric.
$\Longrightarrow A^T = -A$.

Let $A = (a_{ij})_{n\times n}$

$$A^T = (a_{ji}) = -A = -(a_{ij}).$$

$$\therefore \boxed{a_{ii} = -a_{ii}} \Longrightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$$

~~30~~/1.4 ~~Let~~ A be $n\times n$-matrix, Let $C = A - A^T \longrightarrow$ skew sym.
$\boxed{B = A + A^T.} \longrightarrow$ symmetric.

1) $B = A + A^T$.

consider $\boxed{B^T} = (A + A^T)^T = A^T + (A^T)^T = A^T + A$
$= A + A^T$.

$$= A + A^T = \boxed{B}$$

so B is symmetric.

2) $C = A - A^T$ (skew symmetric)

$$\boxed{C^T} = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -\boxed{(A - A^T)} = \boxed{-C}$$

$\therefore$ C is skew-symmetric.

Idempotent matrix A: if $\boxed{A^2 = A}$.

25/1.4) let $\boxed{A \text{ be idempotent}}$ matrix, show that $\boxed{(I-A)}$ is also idempotent?

$$(I-A)^2 = (I-A)(I-A) \overset{?}{=} (I-A)$$

$$= I - A - A + A^2 = I - A - A + A \quad (\text{since } A^2 = A)$$

$$= I - A$$

so $(I-A)$ is idempotent.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A.$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^2 = B \quad (\text{idempotent})$$

$B^2 = B$ (idempotent)

Elementary matrices!

type I
type II
type III

* nonsingular and E^{-1} same type as E.

* If A is $n\times n$-matrix, E_1 elementary.

$B = E_1 A$ $\{ A \xrightarrow[\text{operation}]{\text{row}} B \}$

$C = E_2 B = E_2 E_1 A$ $\{ A \xrightarrow[\text{operation}]{\text{row}} B = E_1 A \xrightarrow[\text{operation}]{\text{row}} C \}$

Def: Let A, B be $n\times n$-matrices, we say B is row equivalent to A if there are elementary matrices $E_1, E_2, \dots, E_k$ such that $B = E_k \cdots E_2 E_1 A$

{B can be obtained from A by applying row operations}

Ex. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix} \xrightarrow{E}$, $B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix} \xrightarrow[-R_3+R_2]{F}$, $C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$.

a) Is B row equivalent to A? (b) Is C row equivalent to A.

$A \xrightarrow[1\cdot R_1 + R_3]{\text{row op.'s}} B$

$\therefore B = EA$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dots \end{pmatrix}$

$\therefore \underline{B = EA}$, $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $\xleftarrow{R_1+R_3}$ $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Find a matrix E such that $B = EA$.

(b) Find elementary matrices E, F such that $\underline{C = FEA}$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}.$$

$\underline{C = FEA}$, $\quad \underline{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\underline{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{-R_3+R_2} I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C row eq. to A

$\boxed{A \cong B.}$

1) If B is row equivalent to A, then $\underline{A \text{ is row equivalent to } B}$.

proof: Assume $B \cong A \Longrightarrow$ there are elementary matrices $E_1, \dots, E_k$ such that $\boxed{B = E_k \cdots E_2 E_1 A}$ ——— $(*)$

$A \xrightarrow{\text{row operations}} \cdots \rightarrow B$

E_k nonsingular: $E_k^{-1}(*) \Rightarrow E_k^{-1} B = E_{k-1} \cdots E_2 E_1 A.$

E_{k-1} " $\Rightarrow E_{k-1}^{-1} E_k^{-1} B = \underline{E_{k-2} \cdots E_2 E_1 A}.$

$\vdots$

$E_1^{-1} E_2^{-1} \cdots E_k^{-1} B = A.$

$$E_1^{-1}E_2^{-1}\cdots E_k^{-1}B = A.$$

$A = E_1^{-1}\cdots E_k^{-1}\boxed{B}$, E_i^{-1} is elementary.

$\therefore$ A is row equivalent to B.

($\boxed{A, B \text{ are row equivalent}}$)

$A \dashrightarrow B \dashrightarrow C$

2) If $B \cong A$ and $C \cong B$, then $C \cong A$.

Proof: Assume $\boxed{B \cong A}$ and $C \cong B$.

$\Downarrow$ there are elem. matrices $E_1, \ldots, E_k$ s.t.

$$\boxed{B = E_k \cdots E_2 E_1 A}$$

$\Downarrow$ there are elem. matrices $F_1, F_2, \ldots, F_r$. s.t.

$$C = F_r \cdots F_2 F_1 B$$

$$C = F_r \cdots F_2 F_1 \boxed{B} = F_r \cdots F_2 F_1 E_k \cdots E_2 E_1 A.$$

$\therefore$ $C \cong A$.

Theorem: Let A be $n\times n$-matrix, then the following statements are equivalent

1) A is nonsingular
2) The system $\boxed{Ax=0}$ has only the zero solution.
3) A is row equivalent to I_n. $\left\{ I_n = E_k \cdots E_1 A \right.$ E_i are elementary. $\left. \right\}$

proof: $\boxed{(1) \Rightarrow (2)}$ $\boxed{(2) \Rightarrow (3)}$ & $\boxed{(3) \Rightarrow (1)}$

proof: $\boxed{(1)\Rightarrow(2)}$ $\boxed{(2)\Rightarrow(3)}$ & $\boxed{(3)\Rightarrow(1)}$

L_i are elementary.

Assume A is nonsingular { A^{-1} exists }.

consider $Ax=0$

if y is a solution to $Ax=0$

$\Rightarrow$ $Ay=0$.

$\xrightarrow{A^{-1}:}$ $A^{-1}Ay=A^{-1}0$

$\rightarrow$ $\boxed{y=0}$

so $Ax=0$ has only the zero solution.

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 1 \\ 1 & -5 & 2 \end{pmatrix}$$

$\boxed{\text{nonsingular?}}$

solve $\boxed{Ax=0}$

only zero sol.

$\Downarrow$

A is nonsingular

Find A^{-1}?

inf # of sol.

$\Downarrow$

$\boxed{A \text{ is sing.}}$

$\boxed{(2)\Rightarrow(3)}$ Assume $Ax=0$ has only the zero solution.

(show $A\simeq I$)

$Ax=0$

G.E.M.

$(A|0) \xrightarrow{\text{reduction}} (U|0)$

{ $U=I$.

Let U be the reduced row echelon form of A.

We know ① $\boxed{A\simeq U.}$

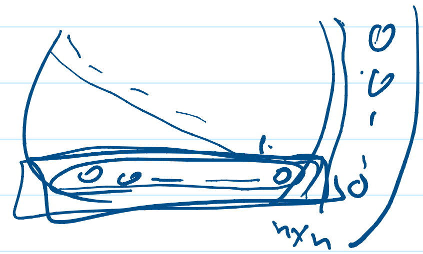
② $Ux=0$ and $Ax=0$ have the same solutions.

so $Ux=0$ has only the zero solution.

so $Ux=0$ has no free variables

If U has a row of zeros, then $Ux=0$ has free variables (not possible).

$$\Rightarrow U=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



, so $U=I$.

so $\boxed{A\simeq I}$

so $\boxed{A \cong I}$

$\boxed{(3) \Rightarrow (1)}$ Assume $A \cong I$. (show A is nonsingular).

$\Rightarrow$ there are elementary matrices $E_1, \ldots, E_k$ s.t.

$$A = E_k \cdots E_2 E_1 I = E_k \cdots E_2 E_1.$$

each E_i is nonsingular $\Rightarrow A = E_k \cdots E_2 E_1$

is nonsingular (a product of nonsingular matrices),

and $A^{-1} = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

Mond 29/3 $\longrightarrow$ 1.1 — 1.4. | Quiz 1 1.1 — 1.4

In class (— — —)