Wednesday, March 24, 2021 9:54 AM

Theorem: If $A_{n\times n}$ is a matrix.

1) A is nonsingular ⟺ ② $Ax=0$ has only the zero solution ⟺ ③ $A \simeq I$.

Ex: Is $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ Is A nonsingular.

solve $Ax=0$ by G.E.M.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = \alpha$ free,

$\therefore$ $Ax=0$ has nonzero solutions (inf # of solutions)

$\therefore$ A is singular.

Remark: Given $A_{n\times n}$, solve $Ax=0$ → has only the zero solution ⟹ A is nonsingular, A^{-1}?

or → has inf. # of solutions ⟹ A is singular.

* Method to find A^{-1} (if exists).

Given $A_{n\times n}$: $A \xrightarrow{\text{reduction}}$ → I : A nonsingular. R.R.E.F.

→ $\neq I$: A singular (a row of zeros)

(a row of zeros)

If $A \cong I$ {A is nonsingular} find A^{-1}?

{A^{-1} exists}

$\Rightarrow$ $I = E_k \cdots E_2 E_1 A$ —— (1)

(1) A^{-1} $\Rightarrow$ $A^{-1} = E_k \cdots E_2 E_1 I$ —— (2)

Method: start $(A \mid I) \xrightarrow{\text{reduction}}$ $(I \mid A^{-1})$ A is nonsingular

$(\neq I \mid X)$ A is singular.

Ex: If $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$. Is A nonsingular, if yes find A^{-1}.

Start: $\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R_1+R_3]{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & \boxed{2} & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$

$\xrightarrow[3R_2+R_3]{-2R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \xrightarrow[-\frac{1}{2}R_3+R_2]{\frac{1}{2}R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right)$

$\xrightarrow[\frac{1}{6}R_3]{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right)$.

$\therefore$ A is nonsingular, $A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$.

Ex: $A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 2 & 3 & 5 \end{pmatrix}$ Is A nonsingular, if yes find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 7 & 7 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{7R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 5 & 7 & 1 \end{array}\right)$$

$\therefore$ A is singular

* $A_{n\times n}$ nonsingular $\Longleftrightarrow$ $Ax=0$ has only the zero solution.

* $A_{n\times n}$ nonsingular $\overset{?}{\Longleftrightarrow}$ $Ax=b$ $\qquad$ $Ax=b$

$Ax=0$ $\rightarrow$ only the zero solution
$\qquad$ or $\rightarrow$ inf # of solution

$Ax=b$ $\rightarrow$ No solution (inconsistent)
$\qquad\rightarrow$ unique solution
$\qquad\rightarrow$ inf # of solution $\Big\}$ consistent.

* $A_{n\times n}$ nonsingular $\Longleftrightarrow$ $Ax=b$

A nonsingular $\Longleftrightarrow$ $Ax=0$ has only the zero solution

Th. Let $A_{n\times n}$ be a matrix. Then A is nonsingular

Th. Let $A_{n\times n}$ be a matrix. Then A is nonsingular if and only if ($\Longleftrightarrow$) the system $\boxed{Ax=b}$ has a unique solution {only one solution}.

Proof: ($\Rightarrow$) Assume A is $\boxed{\text{nonsingular}}$ (A^{-1} exists).

consider $Ax=b$ —— (1)

multiply by A^{-1} from left: $A^{-1}Ax = A^{-1}b$

$\Longrightarrow x = A^{-1}b$

so $Ax=b$ has a unique solution which is $\boxed{x=A^{-1}b}$

($\Leftarrow$) Assume $\boxed{Ax=b}$ has a unique solution $\boxed{w}$. (show A is nonsingular)

$\boxed{\text{and assume } A \text{ is singular.}}$

so $Ax=0$ has nonzero solutions (inf. # of solutions)

let $\boxed{z}$ be a nonzero solution for $\boxed{Ax=0.}$

* $\boxed{Az=0}$ and $\boxed{Aw=b}$, $z\neq 0$.

let $u=z+w$, $\boxed{u\neq w}$ since $z\neq 0$.

Now: $Au = A(z+w) = Az + Aw = 0 + b = b$

$\therefore u$ is a solution for $\boxed{Ax=b.}$

so $Ax=b$ has more than one solution

a contradiction.

So A is nonsingular.

Remarks: ① $A_{n\times n}$ is nonsingular $\Longleftrightarrow$ $\boxed{Ax=0}$ has only the zero solution

② $A_{n\times n}$ is singular $\Longleftrightarrow$ $Ax=0$ has inf. # of solutions (has nonzero solutions).

③ $A_{n\times n}$ is nonsingular $\Longleftrightarrow$ $Ax=b$ $\boxed{\text{has a unique solution.}}$

④ $A_{n\times n}$ is singular $\Longleftrightarrow$ $Ax=b$ has either no solution or inf. number of solutions.

Ex: solve the system

$$\boxed{\begin{aligned} x_1+4x_2+3x_3&=12\\ -x_1-2x_2&=-12\\ 2x_1+2x_2+3x_3&=18. \end{aligned}} \quad (*) \qquad Ax=b.$$

$A^{-1}=$ ✓

$$A=\begin{pmatrix} 1 & 4 & 3\\ -1 & -2 & 0\\ 2 & 2 & 3 \end{pmatrix}$$

by previous example

A is nonsingular.

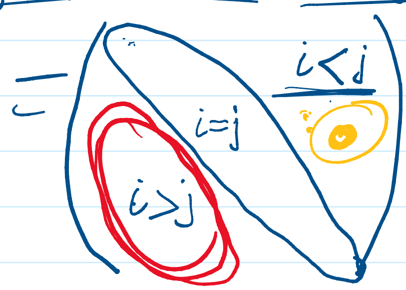
$\Longrightarrow$ $(*)$ has only one solution.

and the solution is $\boxed{x=A^{-1}b}$

$$x=\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 12\\ -12\\ 18 \end{pmatrix} = \begin{pmatrix} 4\\ 4\\ -8/3 \end{pmatrix}$$

Diagonal and triangular Matrices.

$A_{n\times n} =$



a_{ij}.

Def: An $n\times n$-matrix A is called <u>upper triangular</u> if $a_{ij} = 0$ for all $i > j$.

- $A_{n\times n}$ is called <u>lower triangular</u> if $a_{ij} = 0$ for all $i < j$.
- $A_{n\times n}$ is called triangular if A is upper <u>or</u> lower triangular.
- A is called diagonal matrix if $\boxed{a_{ij} = 0}$ for all <u>$i \neq j$</u>.

Ex: $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

- <u>upper triangular.</u>
- not lower triangular.
- triangular.
- not Diagonal matrix.

$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

<u>diagonal.</u>

<u>lower</u> <u>and</u> <u>upper triangular.</u>

triangular.