Wednesday, March 31, 2021 9:59 AM

Chapter 2: Determinant of a matrix.

$\underset{n\times n}{A}$: Is A nonsingular?

$\underset{n\times n}{A}$ nonsingular $\Longleftrightarrow$ $\boxed{A \cong I}$ $(A \xrightarrow{\text{reduction}} \cdots \to I)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

1) $\underset{1\times 1}{A} = (a_{11})$ nonsingular $\Longleftrightarrow$ $a_{11} \neq 0$

real number: $\boxed{a\left(\frac{1}{a}\right) = 1}$ $\boxed{a \neq 0}$

let determinant of A $= \det(A) = a_{11}$

$$\boxed{\therefore \underset{1\times 1}{A} \text{ is nonsingular} \Longleftrightarrow \det(A) \neq 0.}$$

Ex, $A = (5)_{1\times 1}$, $\det(A) = 5 \neq 0$

$\Rightarrow A = (5)$ is nonsingular.

$A^{-1} = \left(\frac{1}{5}\right)$ / $(5)\cdot\left(\frac{1}{5}\right) = (1) = I_{1\times 1}$

$A = (0)$, $\det(A) = 0 \Rightarrow A = (0)$ is singular.

2) $\underset{2\times 2}{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\xrightarrow{?} I$

$A \cong I \Longleftrightarrow a_{11}a_{22} - a_{21}a_{12} \neq 0.$

$\underline{A \cong I} \iff \boxed{a_{11}a_{22} - a_{21}a_{12}} \neq 0.$

A is nonsingular $\iff \boxed{a_{11}a_{22} - a_{12}a_{21}} \neq 0.$

Def: $\det(A_{2\times 2}) = \boxed{a_{11}a_{22} - a_{12}a_{21}}.$

$\therefore$ A is nonsingular $\iff \boxed{\det(A) \neq 0.}$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

Ex: ① $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det(A) = 4 - 6 = -2 \neq 0$

$\therefore$ $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is nonsingular.

and $A^{-1} = \dfrac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{pmatrix}$

② $A = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$, $\det(A) = 12 - 12 = 0.$

$\therefore$ $A = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ is singular. (No inverse)

③ $A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$A \cong I \iff \boxed{a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}} \neq 0$

Def: $\det(A_{3\times 3}) =$

$_{3\times 3})$

$\therefore A$ is nonsingular $\Longleftrightarrow \boxed{\det(A)} \neq 0$

* $A_{4\times 4}$: $a_{14}a_{2\,?}a_{3\,?}a_{41} + \text{———} + \cdots + \text{———}$

24-term.

Def: Let $A = (a_{ij})_{n\times n}$.

we define the $\boxed{\text{cofactor } A_{ij}}$ of a_{ij} as

$A_{ij} = (-1)^{i+j} \det(M_{ij})$, where M_{ij} is the

$(n-1)\times(n-1)$-matrix obtained from A by deleting the ith row

and jth colum. { M_{ij} is called minor of a_{ij} }.

Ex: $A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\boxed{\text{cofactor of } a_{32} \quad A_{32}} = (-1)^5 \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$

$= -(a_{11}a_{23} - a_{21}a_{13}).$

$A_{11} = (-1)^2 \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}_{2\times 2} = a_{22}a_{33} - a_{32}a_{23}.$

$A_{12} = (-1)^3 \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

$A_{13} = (-1)^4 \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = a_{21}a_{32} - a_{31}a_{22}$

* Consider $\boxed{a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{31}a_{23} - a_{21}a_{33})$

* Consider $\boxed{a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{31}a_{23} - a_{21}a_{33})$

$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.$$

$\therefore$ $A_{3\times 3}$: $\boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.}$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 1 & -1 & 4 \end{pmatrix}$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1\times(-1)^2 \begin{vmatrix} 0 & 1 \\ -1 & 4 \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$+ (3)(-1)^4 \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1 + 2(-1)(-9) + (3)(1)(2)$$

$$= \boxed{25} \neq 0$$

$\therefore$ $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 1 & -1 & 4 \end{pmatrix}$ is nonsingular. $\left\{ \boxed{\text{find } A^{-1}\text{?}} \right.$ No answer.

* $A_{3\times 3} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = \det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad \text{(1st row)}$$

$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \quad \text{(2nd row)}$$

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \quad \text{(3rd row).}$$

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \quad \text{(3rd row).}$$

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \quad \text{(1st column)}$$

$$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \quad \text{(2nd column)}$$

$$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

$A_{n\times n} = (a_{ij})_{n\times n}$. we define

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}. \quad \text{(1st row).}$$

$$\boxed{\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}. \quad \text{(ith row)}}$$

cofactor expansion of det(A) in terms of ith row.

or

$$\boxed{\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}.} \quad \text{(jth column)}$$

cofactor expansion of det(A) in terms of jth column.

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 1 & -1 & \boxed{4} \end{pmatrix}$, find det(A) in terms of 3rd column.

$$\det(A) = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

$$= 3(-1)^4 \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} + (1)(-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 4(-1)^6 \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}$$

$$= (3)(2) - (-3) + (4)(4) = 6 + 3 + 16 = \underline{\underline{25.}}$$

Ex. $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ \dots \end{pmatrix}$ find det(A)

Ex: $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ -1 & 1 & 2 & 4 \end{pmatrix}$ 4×4. find $\det(A)$.

nonsingular.

$$\det(A) = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43}$$

$$= 0 + 0 + 0 + 2(-1)^7 \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

3×3

$$= -2\left[1(+)\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + (-1)(-)\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 0\right]$$

$$= -2[1 + 7] = -16 \neq 0$$

Cost in terms of 2×2-determinants

$3\times 3 \longrightarrow 3$ of size 2×2.

$4\times 4 \longrightarrow 4$ of size 3×3.
$\longrightarrow \boxed{12}$ of size 2×2.

$5\times 5 \longrightarrow 5$ of size 4×4
$\longrightarrow 60$ of size 2×2.

Properties: ① Theorem: If A is $n\times n$-matrix, then

$$\det(A) = \det(A^T).$$

② If A is $n\times n$-triangular matrix, then

② If A is $n \times n$ triangular matrix, then
$\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$ (the product of diagonal elements of A).

Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{pmatrix}$

$$\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} = (1)(3)(-5)$$

Ex: $A = \begin{pmatrix} 2 & 1 & 3 & -5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

$$\det(A) = (2)(2)(-3)(6)$$

③ If $A_{n \times n}$ has a row (or column) of zeros, then
$\det(A) = 0$

$A = \begin{pmatrix} & & & \\ 0 & 0 & \cdots & 0 \\ & & & \end{pmatrix}$ | $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \\ 1 & 0 & -5 \end{pmatrix}$

$\det(A) = 0$

④ If $A_{n \times n}$ has two identical rows (or two

(4) If $A_{n \times n}$ has two identical rows (or two identical columns), then $\det(A) = 0$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -5 & 6 \\ 1 & 2 & 3 \end{pmatrix} \implies \det(A) = 0.$$

singular.

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 5 & 2 \\ 3 & -1 & 3 \end{pmatrix}, \quad \det(A) = 0.$$

singular.