Monday, April 5, 2021 10:02 AM

$$\det(A_{n\times n}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} \quad (i\text{th row})$$
$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}. \quad (j\text{th column}).$$

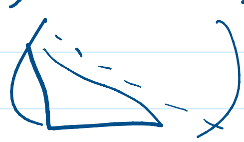
* $\det(A^T) = \det(A)$
* If A is triangular (upper or lower), then
$$\boxed{\det(A) = a_{11}a_{22}\cdots a_{nn}}$$ (product of elements on main diagonal of A)
* If A has a row (or column) of zeros, then $\det(A) = 0$
* If A has two identical rows (or two identical columns), then $\det(A) = 0$.

2.2) properties of det(A).

$$A_{n\times n} \xrightarrow[\text{I, II, III}]{\text{row operations}} U \quad (\text{upper triangular})$$

$\det(U)$ is easy

$$\boxed{\det(A) \xrightarrow{?} \det(U)}$$



Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}$.

$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det(A)$

element of row 1.

$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ — element of row 1, cofactors of row 3.

$$= 1\cdot(-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 2\cdot(-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 7 - 10 + 3 = 0$$

Th. Let A be $n\times n$-matrix, then

$$a_{i1}A_{s1} + a_{i2}A_{s2} + \cdots + a_{in}A_{sn} = \begin{cases} \det(A) & s = i \\ 0 & s \neq i \end{cases}$$

effect of row operations on $\det(A)$.

$$A_{n\times n} \xrightarrow{\text{row operation}} EA = B$$

$$\boxed{\det(EA) \overset{?}{\leadsto} \det(A).}$$

① Case (Row operation II) $\{\alpha R_i\}$ — type I

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\text{ope. II}]{\text{row}} EA = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(EA) = \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in}.$$

$$= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in})$$

$$= \alpha\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\right)$$

$$= \boxed{\alpha \det(A)}, \quad \boxed{E \text{ is of type II}}$$

* $\det(E) = \det(EI) = \alpha \det(I) = \alpha$ $\qquad I \xrightarrow{\alpha R_i} E.$

$$\boxed{\det(EA) = \alpha \det(A) = \det(E)\det(A)}$$

E is of type II $\{\alpha R_i\}$.

Ex. $A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix} \longrightarrow \det(A) = -16.$

① Find $\det(B)$, $B = \begin{pmatrix} 2 & 5 & 4 \\ 9 & 3 & 6 \\ 5 & 4 & 6 \end{pmatrix}$ $\qquad A \xrightarrow{3R_2} B.$

$$\det(B) = 3\det(A) = 3(-16) = -48$$

② Find $\det(C)$, $C = \begin{pmatrix} 4 & 10 & 8 \\ 6 & 2 & 4 \\ 15 & 12 & 18 \end{pmatrix}$.

$$A \xrightarrow{2R_1} \xrightarrow{2R_2} \xrightarrow{3R_3} C$$

$$\det(C) = 3 \times 2 \times 2 \det(A) = 12 \det(A) = 12(-16)$$

* Row operation III: $(cR_i + R_j)$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{pmatrix} \xrightarrow{cR_i + R_j} \overset{B}{\overset{\shortparallel}{EA}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ ca_{i1}+a_{j1} & ca_{i2}+a_{j2} & \cdots & ca_{in}+a_{jn} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow{cR_i + R_j} EA = \begin{pmatrix} \cdots \\ ca_{i1}+a_{j1} & ca_{i2}+a_{j2} & \cdots & ca_{in}+a_{jn} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

E is of type III

$$\det(EA) = (ca_{i1}+a_{j1})B_{j1} + (ca_{i2}+a_{j2})B_{j2} + \cdots + (ca_{in}+a_{jn})B_{jn}$$
$$= (ca_{i1}+a_{j1})A_{j1} + (ca_{i2}+a_{j2})A_{j2} + \cdots + (ca_{in}+a_{jn})A_{jn}$$
$$= c(a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}) + a_{j1}A_{j1} + a_{j2}A_{j2} + \cdots + a_{jn}A_{jn}$$
$$= c(0) + \det(A) = \det(A).$$

* $\boxed{\det(E)} = \det(EI) = \det(I) = 1.$ { E is of type III }

$$\boxed{\det(EA) = \det(A) = \det(E)\det(A)}, \quad E \text{ type III}$$

* Row operation I. ($R_i \leftrightarrow R_j$)

$A \xrightarrow{\text{row oper I}} EA$, E is of type I.

* $\det(EA) = -\det(A)$

$\det(E) = -1$

$$\boxed{\det(EA) = \det(E)\det(A) = -\det(A).}$$

* If A is $n\times n$-matrix, and E is elementary, then

$$\det(EA) = \boxed{\det(E)}\,\det(A).$$

where
$$\det(E) = \begin{cases} -1 & \text{if } E \text{ of type I} \\ \alpha & \text{if } E \text{ of type II} \\ 1 & \text{if } E \text{ of type III} \end{cases}$$

Method to calculate $\det(A)$.

Ex: Find

$$\begin{vmatrix}1&3&4\\2&-1&1\\3&-2&5\end{vmatrix} \underset{-2R_1+R_2}{=} \begin{vmatrix}1&3&4\\0&-7&-7\\3&-2&5\end{vmatrix} \underset{-3R_1+R_3}{=} \begin{vmatrix}1&3&4\\0&\boxed{\begin{matrix}-7&-7\\-11&-7\end{matrix}}\end{vmatrix} = 1(-1)^2\begin{vmatrix}-7&-7\\-11&-7\end{vmatrix}$$

$$= 49 - 77$$
$$= -28.$$

$$\underset{-\frac{11}{7}R_2+R_3}{=} \begin{vmatrix}1&3&4\\0&-7&-7\\0&0&4\end{vmatrix} = -28$$

Ex:

$$\begin{vmatrix}1&2&-1&1\\2&-1&1&0\\1&-1&2&1\\-1&1&0&1\end{vmatrix} = \begin{vmatrix}1&2&-1&1\\0&-5&3&-2\\0&-3&3&0\\0&3&-1&2\end{vmatrix}$$

$$= -\begin{vmatrix}1&2&-1&1\\0&\boxed{-3}&3&0\\0&-5&3&-2\\0&3&-1&2\end{vmatrix} = -(?)\begin{vmatrix}1&2&-1&1\\0&1&-1&0\\0&-5&3&-2\\0&3&-1&2\end{vmatrix}$$

$? = \left(-\frac{1}{3}\right)$ or (-3) ?

$\begin{pmatrix}(3)\\(-3)\end{pmatrix}$?