Wednesday, April 7, 2021 10:00 AM

Sunday: 11/4 . 8:00 — 9:30

Short Exam (8:30 — 9:00) 30 minutes

* ITC: section

* Material: ch 1 + 2.1 + 2.2.

* 2x(quiz)

Meta: Midterm

short exams + quiz → 20% / 25%

6/5

(x out of 6. / x out of 5)

5/6 → 5/5

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* $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$. ($i$th row)

$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (jth column)

* $\det(EA) = \det(E)\det(A)$, E is elementary

$$\det(E) = \begin{cases} -1 & E \text{ of type I} \\ \alpha & E \text{ of type II} \\ 1 & E \text{ of type III} \end{cases}$$

* $A \xrightarrow{\text{reduction}} U$

* $A_{n\times n}$ is nonsingular $\iff \det(A) \neq 0$

(A is singular $\iff \det(A) = 0$).

Ex:

$$\begin{vmatrix}1&2&-1&1\\2&-1&1&0\\1&-1&2&1\\-1&1&0&1\end{vmatrix} = \begin{vmatrix}1&2&-1&1\\0&-5&3&-2\\0&-3&3&0\\0&3&-1&2\end{vmatrix}$$

$$= -\begin{vmatrix}1&2&-1&1\\0&-3&3&0\\0&-5&3&-2\\0&3&-1&2\end{vmatrix}$$

R_2+R_3

$-\frac{5}{3}R_2+R_3$

$$= -\begin{vmatrix}1&2&-1&1\\0&-3&3&0\\0&0&-2&-2\\0&0&2&2\end{vmatrix} = +\begin{vmatrix}1&2&-1&1\\0&-3&3&0\\0&0&+2&+2\\0&0&2&2\end{vmatrix} = 0.$$

$$= -\left(1\cdot(-3)\,(-4+4)\right) = 0$$

R_3+R_4

$$= -\begin{vmatrix}1&2&-1&1\\0&-3&3&0\\0&0&-2&-2\\0&0&0&0\end{vmatrix} = 0.$$

Ex:

$$\underbrace{\begin{vmatrix}1&2&-1\\0&3&1\\0&5&2\end{vmatrix}}_{A} = 3\underbrace{\begin{vmatrix}1&2&-1\\0&1&\frac{1}{3}\\0&5&2\end{vmatrix}}_{EA}$$

$\frac{1}{3}R_2$

$$\boxed{\det(EA) = 3\det(A)}$$

Ex: $A_{4\times 4}$, $\det(A) = -3$

$$\det(2A) = 2^4 \det(A).$$

* If $A = (a_{ij})_{n\times n}$, then $\boxed{\det(kA) = k^n \det(A)}$

* $\det(EA) = \det(E)\det(A)$, E is elementary.

$(\det(AE) = \det(A)\det(E))$.

* $\boxed{\det(AB) \overset{?}{=} \det(A)\det(B)}$?

Th. Let A, B be $n\times n$-matrices, then

$$\boxed{\det(AB) = \det(A)\det(B).}$$

Proof: Let A, B be $n\times n$-matrices.

Case 1: If $\boxed{B \text{ is singular.}} \Longrightarrow \boxed{\det(B) = 0}$

$\boxed{(Q\tfrac{18}{1.5}): \text{ If } A_{n\times n} \text{ any matrix, } B_{n\times n} \text{ is singular, then } C = AB \text{ is singular}}$

so by $Q\frac{18}{1.5}$, we have AB is also singular.

$\Longrightarrow \boxed{\det(AB) = 0}$

Now $\det(A)\det(B) = \det(A)\cdot 0 = 0$

$\therefore \boxed{\det(A)\det(B) = \det(AB)} \ (=0)$

Case(2) If B is nonsingular.

$\Longrightarrow B \simeq I \Longrightarrow$ there are elementary

$\Rightarrow B \cong I \Rightarrow$ there are elementary matrices $E_1, E_2, \ldots, E_k$ s.t. $B = E_k \cdots E_2 E_1 I$

$\Rightarrow \boxed{B = E_k \cdots E_2 E_1}$ (product of elem. matrices)

Now $\det(AB) = \det(\boxed{A E_k \cdots E_2} E_1)$ (show $= \det(A)\det(B)$)

$= \det((A E_k \cdots E_3) E_2) \det(E_1)$.

$= \det(A E_k \cdots E_3) \det(E_2) \det(E_1)$.

$\vdots$

$= \det(A) \underbrace{\det(E_k) \cdots \det(E_2) \det(E_1)}$

$\vdots$

$= \det(A) \det(E_k \cdots E_2 E_1)$

$\therefore \det(AB) = \det(A) \det(B)$.

Ex: $\frac{6}{2.2}$ Let A be nonsingular matrix, show that

$$\boxed{\det(A^{-1})} = \frac{1}{\det(A)}.$$

solution: Let A be nonsingular (A^{-1} exists)

$\Rightarrow \boxed{A A^{-1} = I}$ $(= A^{-1} A)$.

$\Rightarrow \det(A A^{-1}) = \det(I)$.

$\Rightarrow \det(A) \det(A^{-1}) = 1$

$\Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$, $\det(A) \neq 0$.

$$\rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}, \quad \det(A) \neq 0.$$

(A is nonsingular).

7) A, B 3x3–matrices, $\det(A) = 4$, $\det(B) = 5$.

2.2

1) $\det(AB) = \det(A)\det(B) = 4 \times 5 = 20$

2) $\det(3A) = 3^3 \det(A) = (27)(4) = 108$.

3) $\det(A^{-1}B) = \boxed{\det(A^{-1})}\det(B) = \frac{1}{\det(A)} \cdot \det(B) = \frac{5}{4}$

4) $\det(3 A^T B^{-1} A^2) = 3^3 \det(A^T)\det(B^{-1})\det(A^2)$

$= (27) \det(A) \frac{1}{\det(B)} \cdot [\det(A)]^2$

$= 27(4) \frac{1}{5} \cdot (4)^2$

$A_{3\times 3}$, $\det(A) = 4$, $B_{4\times 4}$, $\det(B) = 5$.

$\det(A_{3\times3} B_{4\times4})$ not defined.

AB not defined.

$$\det(A_{3\times4} B_{4\times3})_{3\times3} \neq \det(A_{3\times4})\det(B_{4\times3})$$

4/2.2) Find all values of c that makes $\boxed{A \text{ singular.}} \Leftrightarrow \boxed{\det(A)=0}$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

$$\det(A)=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{vmatrix} =$$

$$= 1\cdot\left(16-(c-1)^2\right)$$

$$= 16 - c^2 - 1 + 2c$$

$$= -c^2+2c+15.$$

$$\det(A)=0 \Longleftrightarrow -c^2+2c+15=0$$

$$\Longleftrightarrow c^2-2c-15=0$$

$$(c-5)(c+3)=0.$$

$$\Longleftrightarrow \boxed{c=5 \quad \text{or} \quad c=-3}$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix} \text{ is singular} \Longleftrightarrow c=5, \text{ or } c=-3.$$

$\boxed{2\text{-}3}$ Adjoint.

cofactor $A_{ij} = (-1)^{i+j}|M_{ij}|$

Def: Let $A=(a_{ij})_{n\times n}$ any matrix,

we define the adjoint of A as

$$\mathrm{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \end{pmatrix}^T \quad \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \end{pmatrix}$$

$$\mathrm{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -2 & 2 & 5 \end{pmatrix}$, Find $\mathrm{adj}(A)$.

$A_{11} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1$ $\quad A_{21} = -\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} = 4$ $\quad A_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2$

$A_{12} = -\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -1$ $\quad A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 9$ $\quad A_{32} = -\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5$

$A_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$ $\quad A_{23} = -\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} = -2$ $\quad A_{33} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$$\therefore \mathrm{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} -1 & -1 & 0 \\ 4 & 9 & -2 \\ -2 & -5 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & 4 & -2 \\ -1 & 9 & -5 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A\,\mathrm{adj}(A) = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -2 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 4 & -2 \\ -1 & 9 & -5 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{\det(A) = -1}$$