Monday, April 12, 2021 9:57 AM

short exam: Mark $/10$ $(12/10)$

$\cancel{\det(A)}$ $\det(A) \neq 0$.

$\boxed{\text{If } A^T = A^{-1} \overset{?}{\Longrightarrow} \det(A) = \pm 1.}$

$\Longrightarrow \det(A^T) = \det(A^{-1})$

$\Longrightarrow \det(A) = \frac{1}{\det(A)} \Longrightarrow \det(A)^2 = 1$

$\det(A) = \pm 1$

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$\det(A) = \pm 1 \Longrightarrow A$ is elem ✗

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ not elem, $\det(A) = 1$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\det(A) = -1$, not elem.

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Review:

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2.3) Adjoint: If $A = (a_{ij})_{n \times n}$

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}.$$

cofactors of row 1. → cofactors of row 2.

Ex.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -2 & 2 & 5 \end{pmatrix}, \quad \operatorname{adj}(A) = \begin{pmatrix} -1 & 4 & -2 \\ -1 & 9 & -5 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A \operatorname{adj}(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Th. If $A = (a_{ij})_{n \times n}$, then $\boxed{A \operatorname{adj}(A) = \boxed{\det(A)\, I}.}$

$$A \operatorname{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & 0 & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

$\Big( \det(A) \quad 0 \quad 0 \cdots 0 \Big)$

$a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$

$$= \begin{pmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & \det(A) \end{pmatrix}$$

$\underline{a_{11}}\,\underline{A_{11}} + \underline{a_{12}}\,\underline{A_{12}} + \cdots + \underline{a_{1n}}\,\underline{A_{1n}} = \det(A)$

$a_{11}A_{21} + a_{12}A_{22} + \cdots + a_{1n}A_{2n} = 0$

$$= \det(A)\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots 0 & 1 \end{pmatrix} = \det(A)\, I .$$

so $\boxed{A \,\mathrm{adj}(A) = \det(A)\, I = \mathrm{adj}(A)\, A} \longrightarrow (*)$

* Case 1 If $A$ is nonsingular, ($A^{-1}$ exists, $\underline{\det(A) \neq 0}$).

$(*) \to \dfrac{1}{\det(A)} A \,\mathrm{adj}(A) = I$

$\Longrightarrow \boxed{A\left(\dfrac{1}{\det(A)}\mathrm{adj}(A)\right) = I} \boxed{= \left(\dfrac{1}{\det(A)}\mathrm{adj}(A)\right) A}$

$\Longrightarrow \boxed{A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A)}$ ✓

Ex. $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

1) Find $\mathrm{adj}(A)$.
2) find $\det(A)$.
3) find $A^{-1}$ (if exists).

1) $\underline{\mathrm{adj}(A)}$:

| | | |
|---|---|---|
| $A_{11} = 2$ | $A_{21} = -(-1) = 1$ | $A_{31} = -2$ |
| $A_{12} = -7$ | $A_{22} = 4$ | $A_{32} = 2$ |
| $A_{13} = 4$ | $A_{23} = -3$ | $A_{33} = 1$. |

$\therefore \mathrm{adj}(A) = \begin{pmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{pmatrix}$.

2) $\underline{\det(A)} = 2\times 2 - 7 + 8 = 5 \neq 0$

(first row of $A$) $\times$ (first column of adj(A))

3) $A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A) = \dfrac{1}{5}\begin{pmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{pmatrix}$

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* If $\boxed{\underset{n\times n}{A}x = b}$ is $\boxed{n\times n\text{- system}, \ A \text{ is nonsingular.}}$

$\Rightarrow Ax = b$ has a unique solution.

Method | sol solution | $x = A^{-1}b$ | { or Use G.Elem. method.

$\Rightarrow Ax=b$ has a unique solution.

Method $\boxed{\text{and solution } \boxed{x = A^{-1}b}}$ or $\boxed{\text{Use G.Elim. method}}$

- $A \sim$
- $Ax \sim$
- $A \simeq I$

$\underset{n\times n}{A}$ $x=0$, $A$ nonsingular

$Ax=0$ has only the zero solution.

Method II) Cramer's Rule.

Theorem: If $A=(a_{ij})_{n\times n}$ is nonsingular, and $b\in\mathbb{R}^n$.
then the unique solution of $Ax=b$ is given by

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}}, \quad i=1,\ldots,n$$

where $A_i$ is the matrix obtained from $A$ by replacing the $i$th column of $A$ by $b$.

Ex.) $\boxed{\text{Use Cramer's rule}}$ to solve

$$x_1+2x_2+x_3=5$$
$$2x_1+2x_2+x_3=6$$
$$x_1+2x_2+3x_3=9.$$

$$A=\begin{pmatrix}1&2&1\\2&2&1\\1&2&3\end{pmatrix}, \quad b=\begin{pmatrix}5\\6\\9\end{pmatrix}$$

$$\det(A)=\begin{vmatrix}1&2&1\\2&2&1\\1&2&3\end{vmatrix} = -4 \neq 0 \Rightarrow A \text{ is nonsingular.}$$

solution $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $x_1=\dfrac{\det(A_1)}{\det(A)} = \dfrac{\begin{vmatrix}5&2&1\\6&2&1\\9&2&3\end{vmatrix}}{-4} = \dfrac{-4}{-4} = 1.$

$$x_2=\frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix}1&5&1\\2&6&1\\1&9&3\end{vmatrix}}{-4} = \frac{-4}{-4} = 1.$$

$$x_3=\frac{|A_3|}{|A|} = \frac{\begin{vmatrix}1&2&5\\2&2&6\\1&2&9\end{vmatrix}}{-4} = \frac{-8}{-4} = 2.$$

so solution $x=\begin{pmatrix}1\\1\\2\end{pmatrix}$.

properties: $A$, $\text{adj}(A)$ $\quad \{\ \underbrace{A\,\text{adj}(A) = \det(A)\, I}_{\text{any matrix.}}\ \}$

① If $A$ is singular ($\det(A)=0$), then $\boxed{A\,\text{adj}(A) = O_{n\times n} = \begin{pmatrix}0&\cdots&0\\0&\cdots&0\end{pmatrix}}$

① If $A$ is singular $(\det(A)=0)$, then $\boxed{A\,\mathrm{adj}(A) = O_{n\times n} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}}$

② If $\underset{n\times n}{A}$ is nonsingular, then $\det(\mathrm{adj}(A)) = \det(A)^{n-1}$, $n>1$

Proof: $A$ is nonsingular, $\det(A)\neq 0$.

We know $A\,\mathrm{adj}(A) = \det(A)\, I$.

$\Rightarrow \det(A\,\mathrm{adj}(A)) = \det(\underbrace{\det(A)}_{k} I)$

$\Rightarrow \det(A)\,\det(\mathrm{adj}(A)) = \det(A)^n \underbrace{\det(I)}_{=1}$.

$\det(A)\,\det(\mathrm{adj}(A)) = \det(A)^n$.

$\frac{1}{\det(A)}$: $\boxed{\det(\mathrm{adj}(A)) = \det(A)^{n-1}}$

$\det(kB) = k^n \det(B)$

$\det(kB) = k^n \det(B)$

$\det(A)\neq 0$

③ If $A$ is nonsingular, then $\mathrm{adj}(A)$ is nonsingular, $n>1$

Proof: If $A$ is nonsingular $(\det(A)\neq 0)$.

$\Rightarrow \det(\mathrm{adj}(A)) = [\det(A)]^{(n-1)} \neq 0$ since $\det(A)\neq 0$.

$\therefore$ $\mathrm{adj}(A)$ is nonsingular.

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Application: Coding

Ex: Message: SEND MONEY.

$\boxed{5, 8, 10, 21, 7, 2, 10, 8, 3.}$

A →
B →
⋮
Y
Z

Decoding: (easy). frequency: E → … ; 8 → E

Use of Matrices: original Message: 5, 8, 10, 21, 7, 2, 10, 8, 3.

Choose a matrix $A$ (nonsingular, $\det(A)=\pm 1$) $\boxed{A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix}}$

$\boxed{A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 1 \end{pmatrix}}$

$A^{-1}$: all entries are integers.

$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$

$$\begin{pmatrix} 2 & 0 & -1 \\ -4 & 1 & 1 \end{pmatrix}$$

$A^{-1}$: all entries are integers.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Now orgnl Message: $$\boxed{B = \begin{pmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{pmatrix}}$$

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 31 & 37 & 29 \\ 80 & 83 & 69 \\ 54 & 67 & 50 \end{pmatrix} = C$$

sent Message.

Sent Message: 31, 80, 54, 37, 83, 67, 29, 69, 50.

reciever: $$B = A^{-1} C = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 31 & 37 & 29 \\ 80 & 83 & 69 \\ 54 & 67 & 50 \end{pmatrix} = \begin{pmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{pmatrix}$$

5, 8, 10, 21, 7, 2, 10, 8, 3

S E N D M . Y.