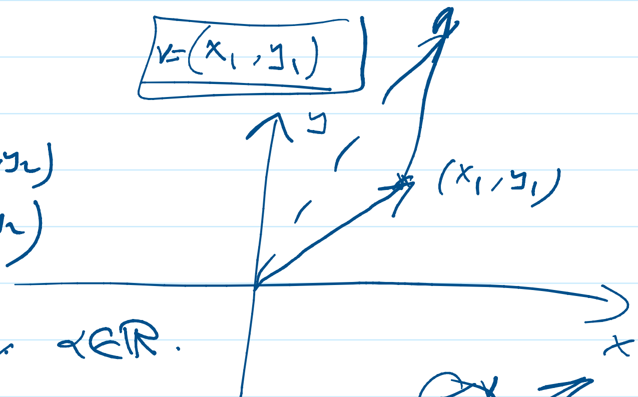
Tuesday, April 13, 2021 8:31 AM

~~Quiz~~ Short exam: M/10 (Review)

Ch3: Vector spaces

Ex: $\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$\boxed{v = (x_1, y_1)}$



vectors: $\oplus$: $v_1 + v_2 = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2, y_1 + y_2)$

scalar multiplication:
$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$, $\alpha \in \mathbb{R}$.

Ex: $M_{2\times 2}(\mathbb{R}) = \left\{ A = (a_{ij})_{2\times 2} : a_{ij} \in \mathbb{R} \right\}$.
$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$.

$\oplus$: $A + B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

$\odot$: $\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$.

Ex. Polynomial: $p(x)$ $q(x)$

$\oplus$: $(p+q)(x)$

$\cdot$: $(\alpha p)(x)$

Def: let V be a set on which two operations are defined, addition and scalar multiplication (i.e. for every $v, w \in V$ we have $\boxed{v + w \in V}$ and for each $v \in V$, α a scalar (real number), we have $\boxed{\alpha v \in V}$. The set V with these operations is called a vector space if the following conditions are satisfied.

1) $v+w = w+v.$ (commutative)

2) $(v+w)+u = v+(w+u)$ (associative)

3) There exists a special element $\boxed{\vec{0} \in V}$ (called zero vector) that satisfies $\boxed{v+\vec{0} = v}$ for all $v \in V$.

4) For every $v \in V$, there exists $-v \in V$ that satisfies
$$v + (-v) = \vec{0}$$

5) $\alpha(v+w) = \alpha v + \alpha w$

6) $(\alpha+\beta)v = \alpha v + \beta v$

7) $(\alpha\beta)v = \alpha(\beta v)$

8) $1v = v$

for all $v, w \in V$, α, β scalars.

Ex: ① $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$ (Euclidean vector space)

under the operations: ① $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} \in \mathbb{R}^n$ standard operations.

② $\alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \in \mathbb{R}^n$ ✓

Is $\mathbb{R}^n$ with these operations a vector space?

1) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ✓

2) ✓

3) $\vec{0} = \boxed{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \in \mathbb{R}^n$ and $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$v + \vec{0} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = v.$$

4) Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, take $-v = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \end{pmatrix}$

4) Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, take $\underline{-v} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$

$$v + (-v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

4), 5), 6), 7), 8)

so $\mathbb{R}^n$ under the standard operation is a vector space.

$$\left.\begin{array}{l} \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \\ \mathbb{R}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\} \end{array}\right\} \text{standard operations}$$

Ex: $\boxed{V} = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ under the operations

$$\begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} y \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} x+y \\ 2 \end{pmatrix} \notin V}$$

$$\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha \end{pmatrix} \notin V$$

$$V \ni \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix} \notin V.$$

$V \ni$

Vector space?

so V under these operations is not a vector space.

Ex: $V = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ under the operations

nonstandard operations.

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} x+y \\ 1 \end{pmatrix} \in V$$

$$\boxed{\alpha \odot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ 1 \end{pmatrix} \in V}$$

Vector space?

1) $\begin{pmatrix} x \\ \, \end{pmatrix} \oplus \begin{pmatrix} y \\ \, \end{pmatrix} = \begin{pmatrix} x+y \\ \, \end{pmatrix}$

1) $\begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} x+y \\ 1 \end{pmatrix}$

$\begin{pmatrix} y \\ 1 \end{pmatrix} \oplus \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} y+x \\ 1 \end{pmatrix}$ = ✓

2) $\left(\begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} y \\ 1 \end{pmatrix}\right) \oplus \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} x+y \\ 1 \end{pmatrix} \oplus \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} (x+y)+z \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \left(\begin{pmatrix} y \\ 1 \end{pmatrix} \oplus \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = \begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} y+z \\ 1 \end{pmatrix} = \begin{pmatrix} x+(y+z) \\ 1 \end{pmatrix} =$

3) zero vector $\vec{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$

and $\boxed{v \oplus \vec{0} = \begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix} = v}$

$\boxed{\vec{0} \in V \text{ and } v + \vec{0} = v}$

so $\boxed{\vec{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

4) let $v = \begin{pmatrix} x \\ 1 \end{pmatrix} \in V$, $\quad -v = \begin{pmatrix} -x \\ 1 \end{pmatrix} \in V$

and $v \oplus (-v) = \begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} -x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{0}$

5) $\alpha(v \oplus w) = \alpha\left(\begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} y \\ 1 \end{pmatrix}\right) = \alpha \cdot \begin{pmatrix} x+y \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(x+y) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x + \alpha y \\ 1 \end{pmatrix}$

$\alpha v \oplus \alpha w = \alpha \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} \oplus \alpha \cdot \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ 1 \end{pmatrix} \oplus \begin{pmatrix} \alpha y \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x + \alpha y \\ 1 \end{pmatrix} =$

6) ✓, 7) ✓, 8)

8) $1 \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}$

$V = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ under these operations is a vector space.

Ex $M_{m\times n}(\mathbb{R}) = \mathbb{R}^{m\times n} = \{A = (a_{ij})_{m\times n} : a_{ij} \in \mathbb{R}\}$.

under operations: $A+B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij}+b_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

$\alpha A = \alpha (a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

is a vector space.

1) $A+B = B+A$ (✓)

2) $(A+B)+C = A+(B+C)$

3) $0 = (0)_{m\times n} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n} \in \mathbb{R}^{m\times n}$

$A + 0 = A$ (✓)

4) Let $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, $-A = (-a_{ij})_{m\times n}$

$A + (-A) = (a_{ij}) + (-a_{ij}) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} = 0$

5), 6), 7), 8) ✓

Ex: $C[a,b]$ the set of all real valued functions that are continuous on $[a,b]$

$C[a,b] = \{f(x) : f \text{ is continuous on } [a,b]\}$

under $(f+g)(x) = f(x) + g(x) \in C[a,b]$

$(\alpha f)(x) = \alpha (f(x)) \in C[a,b]$.

is a vector space.

f cont. and g cont. on $[a,b] \Rightarrow f+g$ is cont. on $[a,b]$

3) 0 (zero function): $0(x) = 0$, $\forall x \in [a,b]$.

③ 0 (zero function) : $0(x) = 0$, $\forall x \in [a, b]$.

$(0+f)(x) = 0(x) + f(x) = 0 + f(x) = f(x)$.

④ Let $f(x) \in C[a,b]$, $(-f)(x) = -f(x)$, $\forall x \in [a,b]$.

$f(x) + (-f(x)) = 0$

5) ——, 8) ✓

Ex: Polynomials

P_n = the set of all polynomial with degree less than n.

$$P_n = \{ p(x) : \deg(p(x)) < n \}.$$

$$= \{ p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R} \},$$

under the operations: Let $\left.\begin{array}{l} p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \\ q(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} \end{array}\right\} \in P_n$.

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1}) x^{n-1} \in P_n$$

$$(\alpha p)(x) = \alpha a_0 + (\alpha a_1) x + \cdots + (\alpha a_{n-1}) x^{n-1} \in P_n.$$

1) ..., ③ 0 : zero polynomial : $0(x) = 0 + 0x + 0x^2 + \cdots + 0x^{n-1}$

$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$

$$(p+0)(x) = (a_0 + 0) + (a_1 + 0)x + \cdots + (a_{n-1} + 0) x^{n-1}$$

$$= a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} = p(x)$$

④ $p(x) \in P_n$: $-p(x) = -a_0 - a_1 x - \cdots - a_{n-1} x^{n-1} \in P_n$.

standard: $\mathbb{R}^n$, P_n, $\mathbb{R}^{m\times n}$, $C[a,b]$. are vector spaces.
(standard operations)

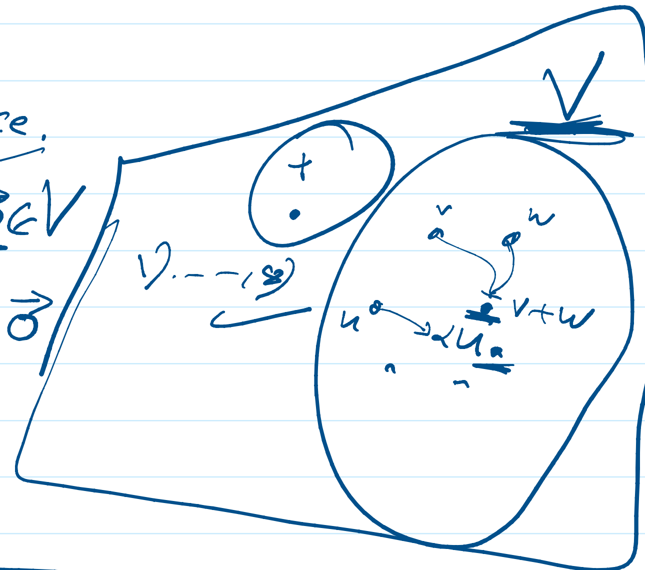
If V is a vector space, and $v\in V$, $0\in\mathbb{R}$

$0\in\mathbb{R}$, $\vec{0}\in V$

$$\boxed{0v = \vec{0}}$$

$$0\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix} = \vec{0}$$

$$0A = O_{m\times n}$$



Theorem: Let $\boxed{V \text{ be a vector space}}$ & $v\in V$, then

1) $0v = \vec{0}$
2) If $u+w=\vec{0}$, then $w=-u$
3) $(-1)v = -v$, $\forall v\in V$.

proof: