Monday, April 19, 2021 9:55 AM

* Vector space. $(V, +, \cdot)$

$(+)$: $v_1, v_2 \in V \Longrightarrow v_1 + v_2 \in V$

$(\cdot)$: $v_1 \in V$, α $\boxed{\text{scalar} \\ \text{real number}}$: $\alpha v_1 \in V$.

1) $u+v = v+u$
2) $u+(v+w) = (u+v)+w$.
⋮
8) $1 \cdot u = u$, for all $u \in V$.

Standard Examples: ① $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$ is a vector space.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix} \in \mathbb{R}^n.$$

② $P_n = \{ p(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} : a_i \in \mathbb{R} \}$ polynomials of degree less than n.

$(+)$: $\boxed{p(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}}$ $q(x) = b_0 + b_1 x + \dots + b_{n-1} x^{n-1}$

$$(p+q)(x) = a_0 + b_0 + (a_1+b_1)x + \dots + (a_{n-1}+b_{n-1})x^{n-1}.$$

$$(\alpha p)(x) = \alpha a_0 + \alpha a_1 x + \dots + (\alpha a_{n-1}) x^{n-1}$$

is a vector space.

③ $\mathbb{R}^{m \times n} = M_{m \times n}(\mathbb{R}) = \{ A = (a_{ij})_{m \times n} : a_{ij} \in \mathbb{R} \}$.

$(+)$: If $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$

(+): If $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{m\times n}$

$$A+B=(a_{ij}+b_{ij})_{m\times n}.$$

(·): $\alpha A=(\alpha a_{ij})$

is a vector space.

(4) $C[a,b]=\{\text{continuous functions on } [a,b]\}$

$$=\{f:[a,b]\longrightarrow\mathbb{R} : f(x) \text{ is cont. on } [a,b]\}.$$

$$(f+g)(x)=f(x)+g(x).$$

$$(\alpha f)(x)=\alpha f(x)$$

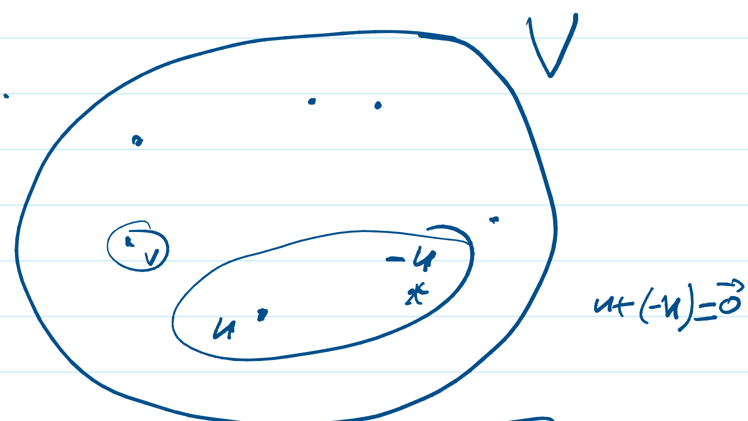
is a vector space.

• zero vector $\vec{0}$

(1) in $\mathbb{R}^n$: $\vec{0}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$

(2) in P_n: $\vec{0}=0+0x+\dots+0x^{n-1}$!

(3) in $M_{m\times n}(\mathbb{R})$: $\vec{0}=\begin{bmatrix}0&\cdots&0\\\vdots&&\vdots\\0&\cdots&0\end{bmatrix}_{m\times n}$.



$\boxed{\vec{0}+v=v}$

$0\in\mathbb{R}$, $v\in V$.

$$0v=\vec{0}$$

If $u+w=\vec{0}$ $\qquad$ $u+(-u)=\vec{0}$

$\therefore\ w=-u.$

Theorem: Let $\boxed{V \text{ be a vector space}}$ & $v \in V$, then

1) $0v = \vec{0}$

2) If $u + w = \vec{0}$, then $w = \boxed{-u}$

3) $(-1)v = -v$, $\forall v \in V$.

{ 8) $1 \cdot v = v$

proof:

① Let $v \in V$ (show $0v = \vec{0}$)

{ we can use conditions $\boxed{(1), \ldots, (8).}$ properties of real numbers }

$(\alpha + \beta)v = \alpha v + \beta v$

$0 + 0 = 0$

$\Rightarrow (0+0)v = 0v$

$\Rightarrow 0v + 0v = 0v$ —— (1)

$0v \in V \Rightarrow -0v \in V$. { cond (4) }

add $-0v$ to both sides of (1)

$\Rightarrow 0v + (0v + (-0v)) = 0v + (-0v)$

$\Rightarrow 0v + \vec{0} = \vec{0}$

$w + \vec{0} = w$ (3)

$\Rightarrow \boxed{0v = \vec{0}}$

② Assume $\boxed{u + w = \vec{0}}$ —— (1) (show $w = -u$).

$u, w \in V \Rightarrow -u \in V$

add $-u$ to both sides of (1)

$\Rightarrow (-u + u) + w = -u + \vec{0}$

$\Rightarrow \vec{0} + w = -u$

$\Rightarrow w = -u$

1) $u + v = v + u$.

4) $u + (-u) = \vec{0}$

3) $w + \vec{0} = w$.

③ $(-1)v = -v$, for any $v \in V$.

we know $1+(-1)=0$ (real numbers).

$\Rightarrow (1+(-1))v = 0v$

$\Rightarrow 1\cdot v + (-1)v = \vec{0}$

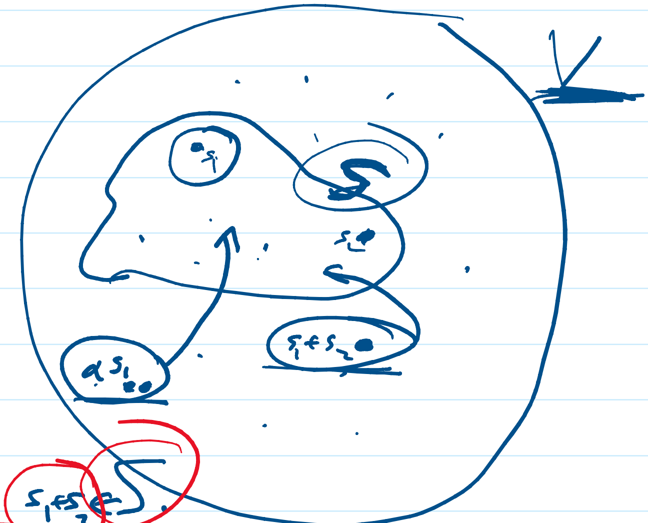
$\Rightarrow v + (-1)v = \vec{0}$

by part (2), $\Rightarrow (-1)v = -v$.

3.2 Subspace:

Def: Let V be a vector space and $S \subseteq V$, $\boxed{S \neq \emptyset}$ (S is nonempty). S is called a subspace of V if

1) For any $s_1, s_2 \in S$ we have $s_1+s_2 \in S$.
2) For any $s_1 \in S$ and $\alpha \in \mathbb{R}$, we have $\alpha s_1 \in S$.

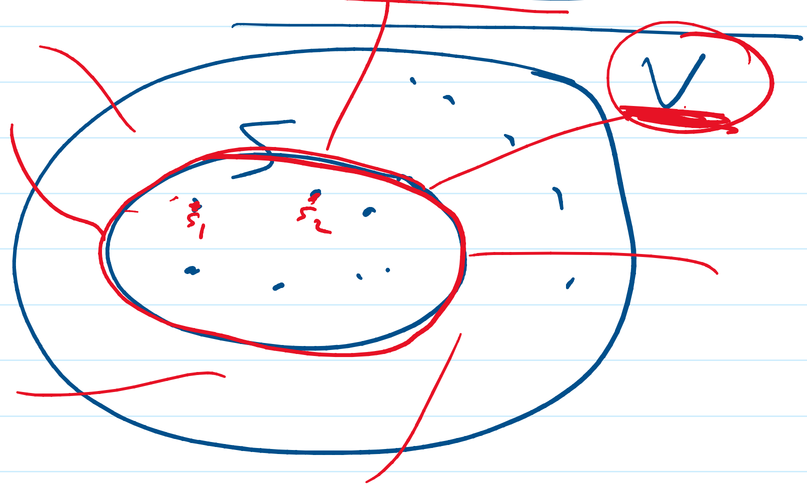


Conditions for S

1) $s_1+s_2 = s_2+s_1$
2) $s_1+(s_2+s_3) = (s_1+s_2)+s_3$

⋮

8) $1 . s = s$



Remark: ① If S is a subspace of V, then S is a vector space.

Ex: Let $S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$. Is S a subspace of $\mathbb{R}^2$ $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ \dots \end{pmatrix}$

Ex: Let $S = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \}$. Is S a subspace of $\mathbb{R}^2$?

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

① $S \neq \emptyset$, $\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S}$ — check if $\vec{0} \in S$. $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$

② Let $s_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \in S$.

Now $s_1 + s_2 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \in S$

③ $s_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in S$, $\alpha \in \mathbb{R}$.

Now $\alpha s_1 = \alpha \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha 0 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix} \in S$

$\therefore S$ is a subspace of $\mathbb{R}^2$.

$\boxed{S = \{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \} \subseteq \mathbb{R}^2 \qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S}$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ — $\mathbb{R}^2$ standard operations.

$5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$

$\oplus$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_2 \\ x_2 + y_1 \end{pmatrix}$ — nonstandard

$\odot$: $\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_2 \\ \alpha x_1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

$\vec{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ($+$), $\vec{0} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ($\oplus$)

Remark: If $\boxed{S \text{ is a subspace of } V}$ then $\boxed{0_V \in S.}$

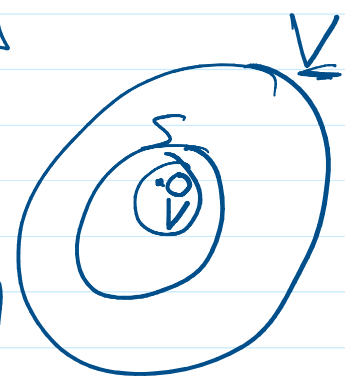
Proof: Let S is a subspace of V.

$\Rightarrow S \neq \emptyset$, let $s \in S$.

and consider $\alpha = 0 \in \mathbb{R}$.

$\boxed{\alpha s = 0s \in S}$ (S is a subspace)

$\Rightarrow$ but $0s = 0_V$, so $\boxed{0_V \in S.}$



$\boxed{\text{If } f(x) \text{ is diff, then } f \text{ is cont.}}$

$\boxed{\text{use: If } f(x) \text{ is not cont, then } f(x) \text{ is not diff.}}$

$f(x)$ is not diff.

$\boxed{\text{Remark: If } V \text{ a vector space, } S \subseteq V. \text{ If } \underset{V}{0} \notin S, \text{ then } S \text{ is not a subspace of } V.}$

Ex. $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : \boxed{a+b=1} \right\}$ Is S a subspace of $\mathbb{R}^2$.

1) $\boxed{S \neq \phi ?}$: $\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S}$ since $0+0=0 \neq 1$.

$\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S$

$\boxed{\therefore S \text{ is not a subspace.}}$

$\begin{pmatrix} 3 \\ -4 \end{pmatrix} \notin S$ $\quad 3+(-4)=-1$

$\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S$

$(2) \; , (3)$ ~~X~~

Ex: $S = \left\{ A = (a_{ij})_{2\times 2} : \det(A) = 0 \right\}$ Is S a subspace of $\boxed{\mathbb{R}^{2\times 2}}$.

1) $S \neq \phi$? ✓ $\boxed{0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S}$ $\left(\det(0_{2\times 2}) = 0 \right)$

2) Let $A, B \in S \Rightarrow \det(A) = 0, \det(B) = 0$.

Is $A+B \in S$?? $\quad | \quad \det(A+B) \overset{?}{=} 0$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S$

but $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin S$

$\therefore S$ is not a subspace of $\mathbb{R}^{2\times 2}$.

Ex: $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a+b=1 \right\}$ Is S a subspace of $\mathbb{R}^2$.

1) $\boxed{S \neq \phi \checkmark}$ since $\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S$ —

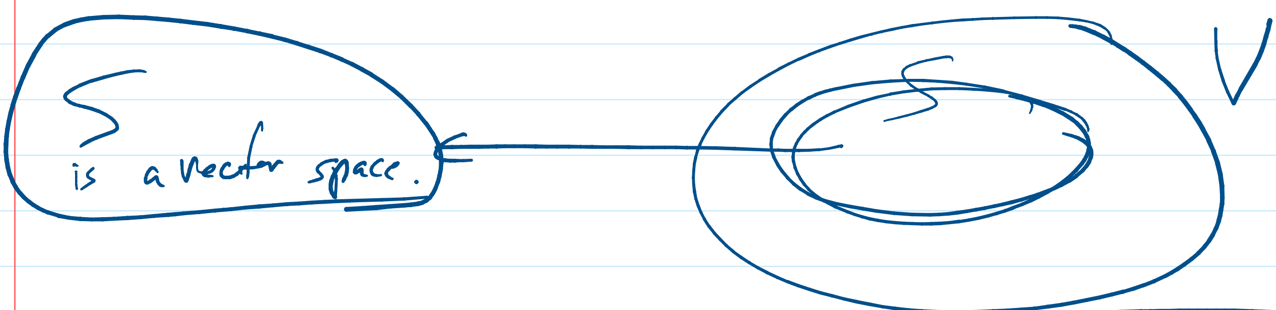
1) $\boxed{S \neq \phi}$ ✓ since $\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S$

2) let $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in S \implies a+b=1, \quad c+d=1.$

Now $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \overset{?}{\notin} S$

consider $a+c+b+d = 1+1 = 2 \neq 1.$? L

take $\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S$, $\begin{pmatrix} -2 \\ 3 \end{pmatrix} \in S$ but $\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \notin S$



Ex: S is the set of all symmetric $n \times n$-matrices, is S a subspace of $\mathbb{R}^{n\times n}$.

$$S = \{A \in \mathbb{R}^{n\times n} : A \text{ is symmetric}\} = \{A \in \mathbb{R}^{n\times n} : A^T = A\}.$$

1) $S \neq \phi$ ✓ $\quad 0 = (0)_{n\times n} \in S \quad 0^T = 0$

2) let $A, B \in S \implies A^T = A, \quad B^T = B.$

Is $A+B \in S$?? : consider $(A+B)^T = A^T + B^T = A+B$

$\therefore \underline{A+B \in S}$

3) let $A \in S \quad \alpha \in \mathbb{R}$

$\implies A^T = A \quad \{A \text{ is symmetric}\}$

Now $\alpha A \overset{?}{\in} S$: $(\alpha A)^T = \alpha(A^T) = \alpha A$

$\Rightarrow \underline{\underline{A = A}} \; (A \ldots)$

Now $\alpha A \overset{?}{\in} S$: $\underline{(\alpha A)^T = \alpha (A^T) = \alpha A}$.

$\therefore$ $\alpha A \in S$.

(1), (2) & (3) $\Rightarrow$ S is a subspace of $\mathbb{R}^{n \times n}$.